To the memory of my darling wife, Margaret

Structural and Stress Analysis

Third Edition

Dr. T.H.G. Megson

Senior Lecturer in Civil Engineering (retired)
University of Leeds

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Preface to the First Edition

The purpose of this book is to provide, in a unified form, a text covering the associated topics of structural and stress analysis for students of civil engineering during the first two years of their degree course. The book is also intended for students studying for Higher National Diplomas, Higher National Certificates and related courses in civil engineering.

Frequently, textbooks on these topics concentrate on structural analysis or stress analysis and often they are lectured as two separate courses. There is, however, a degree of overlap between the two subjects and, moreover, they are closely related. In this book, therefore, they are presented in a unified form which illustrates their interdependence. This is particularly important at the first-year level where there is a tendency for students to 'compartmentalize' subjects so that an overall appreciation of the subject is lost.

The subject matter presented here is confined to the topics students would be expected to study in their first two years since third- and fourth-year courses in structural and/or stress analysis can be relatively highly specialized and are therefore best served by specialist texts. Furthermore, the topics are arranged in a logical manner so that one follows naturally on from another. Thus, for example, internal force systems in statically determinate structures are determined before their associated stresses and strains are considered, while complex stress and strain systems produced by the simultaneous application of different types of load follow the determination of stresses and strains due to the loads acting separately.

Although in practice modern methods of analysis are largely computer based, the methods presented in this book form, in many cases, the basis for the establishment of the flexibility and stiffness matrices that are used in computer-based analysis. It is therefore advantageous for these methods to be studied since, otherwise, the student would not obtain an appreciation of structural behaviour, an essential part of the structural designer's background.

In recent years some students enrolling for degree courses in civil engineering, while being perfectly qualified from the point of view of pure mathematics, lack a knowledge of structural mechanics, an essential basis for the study of structural and stress analysis. Therefore a chapter devoted to those principles of statics that are a necessary preliminary has been included.

As stated above, the topics have been arranged in a logical sequence so that they form a coherent and progressive 'story'. Hence, in Chapter 1, structures are considered in terms of their function, their geometries in different roles, their methods of support and the differences between their statically determinate and indeterminate forms. Also considered is the role of analysis in the design process and methods of idealizing structures so that they become amenable to analysis. In Chapter 2 the necessary principles of statics are discussed and applied directly to the calculation of support reactions. Chapters 3–6 are concerned with the determination of internal force distributions in statically determinate beams, trusses, cables and arches, while in Chapter 7 stress and strain are discussed and stress–strain relationships established. The relationships between the elastic constants are then derived and the concept of strain energy in axial tension and compression introduced. This is then applied to the determination of the effects of impact loads, the calculation of displacements in axially loaded members and the deflection of a simple truss. Subsequently, some simple statically indeterminate systems are analysed and the compatibility of displacement condition introduced. Finally, expressions for the stresses in thin-walled pressure vessels are derived. The properties of the different materials used in civil engineering are investigated in Chapter 8 together with an introduction to the phenomena of strain-hardening, creep and relaxation igue; a table of the properties of the more common civil engineering materials is given at the the chapter. Chapters 9, 10 and 11 are respectively concerned with the stresses produced by ding, shear and torsion of beams while Chapter 12 investigates composite beams. Deflections bending and shear are determined in Chapter 13, which also includes the application of the to the analysis of some statically indeterminate beams. Having determined stress distributions ed by the separate actions of different types of load, we consider, in Chapter 14, the state of nd strain at a point in a structural member when the loads act simultaneously. This leads directly experimental determination of surface strains and stresses and the theories of elastic failure for uctile and brittle materials. Chapter 15 contains a detailed discussion of the principle of virtual nd the various energy methods. These are applied to the determination of the displacements of and trusses and to the determination of the effects of temperature gradients in beams. Finally, iprocal theorems are derived and their use illustrated. Chapter 16 is concerned solely with the of statically indeterminate structures. Initially methods for determining the degree of statical nematic indeterminacy of a structure are described and then the methods presented in r 15 are used to analyse statically indeterminate beams, trusses, braced beams, portal frames and ined arches. Special methods of analysis, i.e. slope–deflection and moment distribution, are pplied to continuous beams and frames. The chapter is concluded by an introduction to matrix ls. Chapter 17 covers influence lines for beams, trusses and continuous beams while Chapter 18 gates the stability of columns.

merous worked examples are presented in the text to illustrate the theory, while a selection of ced problems with answers is given at the end of each chapter.

T.H.G. Megson

Preface to the Second Edition

Since 'Structural and Stress Analysis' was first published changes have taken place in courses leading to degrees and other qualifications in civil and structural engineering. Universities and other institutions of higher education have had to adapt to the different academic backgrounds of their students so that they can no longer assume a basic knowledge of, say, mechanics with the result that courses in structural and stress analysis must begin at a more elementary stage. The second edition of 'Structural and Stress Analysis' is intended to address this issue.

Although the feedback from reviewers of the first edition was generally encouraging there were suggestions for changes in presentation and for the inclusion of topics that had been omitted. This now means, in fact, that while the first edition was originally intended to cover the first two years of a degree scheme, the second edition has been expanded so that it includes third- and fourth-year topics such as the plastic analysis of frames, the finite element method and yield line analysis of slabs. Furthermore, the introductions to the earlier chapters have been extended and in Chapter 1, for example, the discussions of structural loadings, structural forms, structural elements and materials are now more detailed. Chapter 2, which presents the principles of statics, now begins with definitions of force and mass while in Chapter 3 a change in axis system is introduced and the sign convention for shear force reversed.

Chapters 4, 5 and 6, in which the analysis of trusses, cables and arches is presented, remain essentially the same although Chapter 4 has been extended to include an illustration of a computer-based approach.

In Chapter 7, stress and strain, some of the original topics have been omitted; these are some examples on the use of strain energy such as impact loading, suddenly applied loads and the solutions for the deflections of simple structures and the analysis of a statically indeterminate truss which is covered later.

The discussion of the properties of engineering materials in Chapter 8 has been expanded as has the table of material properties given at the end of the chapter.

Chapter 9 on the bending of beams has been modified considerably. The change in axis system and the sign convention for shear force is now included and the discussion of the mechanics of bending more descriptive than previously. The work on the plastic bending of beams has been removed and is now contained in a completely new chapter (18) on plastic analysis. The introduction to Chapter 10 on the shear of beams now contains an illustration of how complementary shear stresses in beams are produced and is also, of course, modified to allow for the change in axis system and sign convention. Chapter 11 on the torsion of beams remains virtually unchanged as does Chapter 12 on composite beams apart from the change in axis system and sign convention. Beam deflections are considered in Chapter 13 which is also modified to accommodate the change in axis system and sign convention.

The analysis of complex stress and strain in Chapter 14 is affected by the change in axis system and also by the change in sign convention for shear force. Mohr's circle for stress and for strain are, for example, completely redrawn.

Chapters 15 and 16, energy methods and the analysis of statically indeterminate structures, are unchanged except that the introduction to matrix methods in Chapter 16 has been expanded and is now part of Chapter 17 which is new and includes the finite element method of analysis.

Chapter 18, as mentioned previously, is devoted to the plastic analysis of beams and frames while Chapter 19 contains yield line theory for the ultimate load analysis of slabs.

hapters 20 and 21, which were Chapters 17 and 18 in the first edition, on influence lines and ural instability respectively, are modified to allow for the change in axis system and, where appro-, for the change in sign convention for shear force.

wo appendices have been added. Appendix A gives a list of the properties of a range of standard sec- while Appendix B gives shear force and bending moment distributions and deflections for standard of beams.

nally, an accompanying Solutions Manual has been produced which gives detailed solutions for all oblems set at the end of each chapter.

T.H.G. Megson

Preface to the Third Edition

After the encouraging response to the second edition, the main features of the third edition are an increase in the number of worked examples and end of chapter problems, together with an extension of the work on fatigue, to the prediction of the fatigue life of a structure in terms of the number of cycles to failure, and to the study of crack propagation and crack propagation rates.

The accompanying Solutions Manual gives detailed solutions for all the problems set at the end of each chapter. It can be found at: http://booksite.elsevier.com/9780080999364

T.H.G. Megson

CHAPTER

Introduction

1

In the past it was common practice to teach structural analysis and stress analysis, or theory of structures and strength of materials as they were frequently known, as two separate subjects where, generally, structural analysis was concerned with the calculation of internal force systems and stress analysis involved the determination of the corresponding internal stresses and associated strains. Inevitably a degree of overlap occurred. For example, the calculation of shear force and bending moment distributions in beams would be presented in both structural and stress analysis courses, as would the determination of displacements. In fact, a knowledge of methods of determining displacements is essential in the analysis of some statically indeterminate structures. It seems logical, therefore, to unify the two subjects so that the 'story' can be told progressively with one topic following naturally on from another.

In this chapter we shall look at the function of a structure and then the different kinds of loads the structures carry. We shall examine some structural systems and ways in which they are supported. We shall also discuss the difference between statically determinate and indeterminate structures and the role of analysis in the design process. Finally, we shall look at ways in which structures and loads can be idealized to make structures easier to analyse.

1.1 Function of a structure

The basic function of any structure is to carry loads and transmit forces. These arise in a variety of ways and depend, generally, upon the purpose for which the structure has been built. For example, in a steel-framed multistorey building the steel frame supports the roof and floors, the external walls or cladding and also resists the action of wind loads. In turn, the external walls provide protection for the interior of the building and transmit wind loads through the floor slabs to the frame while the roof carries snow and wind loads which are also transmitted to the frame. In addition, the floor slabs carry people, furniture, floor coverings, etc. All these loads are transmitted by the steel frame to the foundations of the building on which the structure rests and which form a structural system in their own right.

Other structures carry other types of load. A bridge structure supports a deck which allows the passage of pedestrians and vehicles, dams hold back large volumes of water, retaining walls prevent the slippage of embankments and offshore structures carry drilling rigs, accommodation for their crews, helicopter pads and resist the action of the sea and the elements. Harbour docks and jetties carry cranes for unloading cargo and must resist the impact of docking ships. Petroleum and gas storage tanks must be able to resist internal pressure and, at the same time, possess the strength and stability to carry wind and snow loads. Television transmitting masts are usually extremely tall and placed in elevated positions where wind and snow loads are the major factors. Other structures, such as ships, aircraft, space vehicles, cars, etc. carry equally complex loading systems but fall outside the realm of structural engineering. However, no matter how simple or how complex a structure may be or whether the structure is intended to carry loads or merely act as a protective covering, there will be one load which it will always carry, its own weight.

Loads

:ally, loads on civil engineering structures fall into two categories. *Dead loads* are loads that act on cture all the time and include its self-weight, fixtures, such as service ducts and light fittings, sus-d ceilings, cladding and floor finishes, etc. Interestingly, machinery and computing equipment sumed to be movable even though they may be fixed into position. *Live* or *imposed loads* are mov-r actually moving loads; these include vehicles crossing a bridge, snow, people, temporary parti-and so on. *Wind loads* are live loads but their effects are considered separately because they are :d by the location, size and shape of a structure. Soil or hydrostatic pressure and dynamic effects ced, for example, by vibrating machinery, wind gusts, wave action or even earthquake action in parts of the world, are the other types of load.

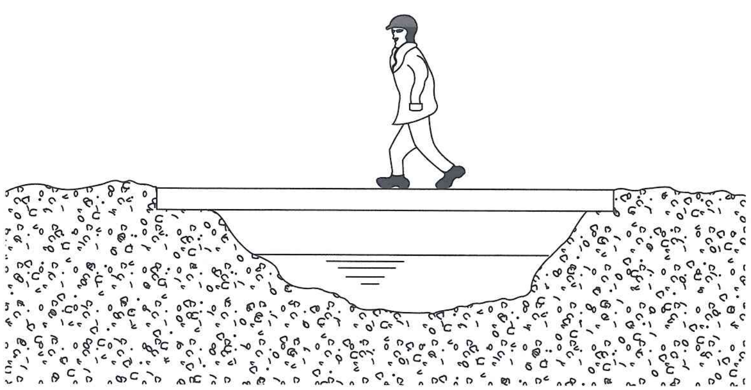
most cases Codes of Practice specify values of the above loads which must be used in design. values, however, are usually multiplied by a *factor of safety* to allow for uncertainties; generally ctors of safety used for live loads tend to be greater than those applied to dead loads because live are more difficult to determine accurately.

Structural systems

ecision as to which type of structural system to use rests with the structural designer whose choice epend on the purpose for which the structure is required, the materials to be used and any aes-considerations that may apply. It is possible that more than one structural system will satisfy the ements of the problem; the designer must then rely on experience and skill to choose the best on. On the other hand there may be scope for a new and novel structure which provides savings t and improvements in appearance.

1s

ural systems are made up of a number of structural elements although it is possible for an element : structure to be a complete structure in its own right. For example, a simple *beam* may be used ry a footpath over a stream (Fig. 1.1) or form part of a multistorey frame (Fig. 1.2). Beams are f the commonest structural elements and carry loads by developing shear forces and bending nts along their length as we shall see in Chapter 3.



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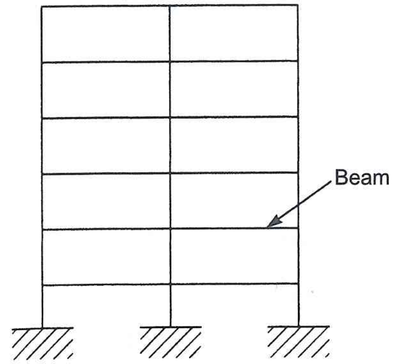


FIGURE 1.2

Beam as a structural element.

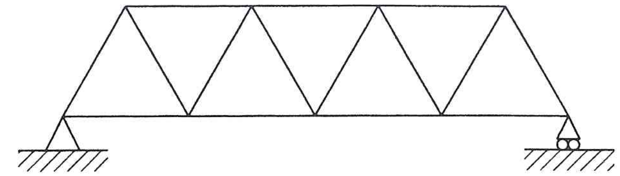


FIGURE 1.3

Warren truss.

Trusses

As spans increase the use of beams to support bridge decks becomes uneconomical. For moderately large spans *trusses* are sometimes used. These are arrangements of straight members connected at their ends. They carry loads by developing axial forces in their members but this is only exactly true if the ends of the members are pinned together, the members form a triangulated system and loads are applied only at the joints (see Section 4.2). Their depth, for the same span and load, will be greater than that of a beam but, because of their skeletal construction, a truss will be lighter. The Warren truss shown in Fig. 1.3 is a two-dimensional *plane truss* and is typical of those used to support bridge decks; other forms are shown in Fig. 4.1.

Trusses are not restricted to two-dimensional systems. Three-dimensional trusses, or *space trusses*, are found where the use of a plane truss would be impracticable. Examples are the bridge deck support system in the Forth Road Bridge and the entrance pyramid of the Louvre in Paris.

Moment frames

Moment frames differ from trusses in that they derive their stability from their joints which are rigid, not pinned. Also their members can carry loads applied along their length which means that internal member forces will generally consist of shear forces and bending moments (see Chapter 3) as well as axial loads although these, in some circumstances, may be negligibly small.

Figure 1.2 shows an example of a two-bay, multistorey moment frame where the horizontal members are beams and the vertical members are called *columns*. Figures 1.4(a) and (b) show examples of *Portal* frames which are used in single storey industrial construction where large, unobstructed working areas are required; for extremely large areas several Portal frames of the type shown in Fig. 1.4(b) are combined to form a multibay system as shown in Fig. 1.5.

Moment frames are comparatively easy to erect since their construction usually involves the connection of steel beams and columns by bolting or welding; for example, the Empire State Building in New York was completed in 18 months.

Arches

The use of trusses to support bridge decks becomes impracticable for longer than moderate spans. In

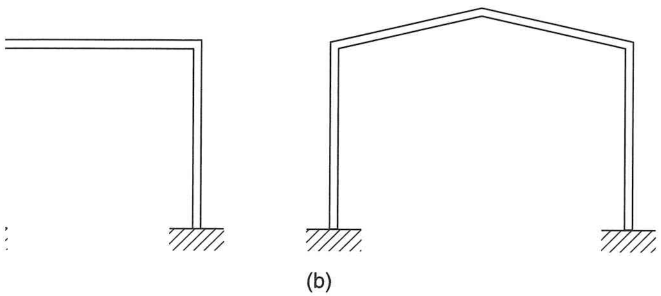


FIGURE 1.4

Portal frames.

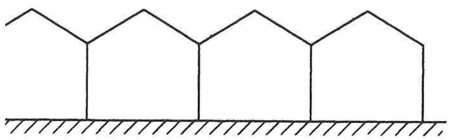
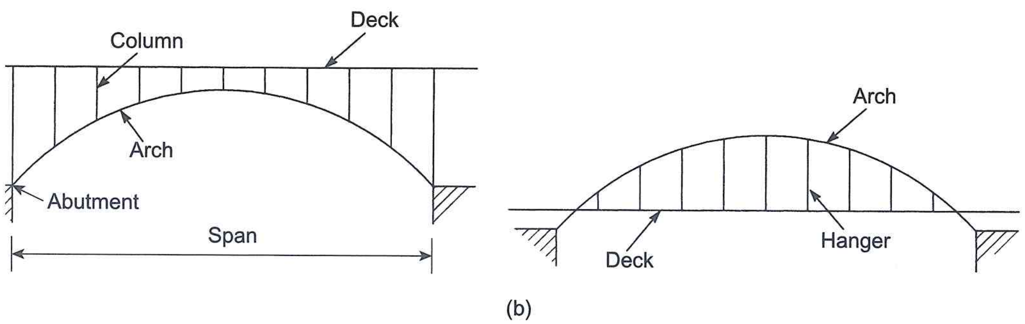


FIGURE 1.5

Multibay single storey building.



E 1.6

·s as bridge deck supports.

ıns supported, in turn, by the arch. Alternatively the bridge deck may be suspended from the arch ıngers, as shown in Fig. 1.6(b). Arches carry most of their loads by developing compressive stresses n the arch itself and therefore in the past were frequently constructed using materials of high com- ve strength and low tensile strength such as masonry. In addition to bridges, arches are used to ›rt roofs. They may be constructed in a variety of geometries; they may be semicircular, parabolic :n linear where the members comprising the arch are straight. The vertical loads on an arch would the ends of the arch to *spread*, in other words the arch would flatten, if it were not for the abut- ; which support its ends in both horizontal and vertical directions. We shall see in Chapter 6 that fect of this horizontal support is to reduce the bending moment in the arch so that for the same ıg and span the cross section of the arch would be much smaller than that of a horizontal beam.

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xceptionally long-span bridges, and sometimes for short spans, cables are used to support the

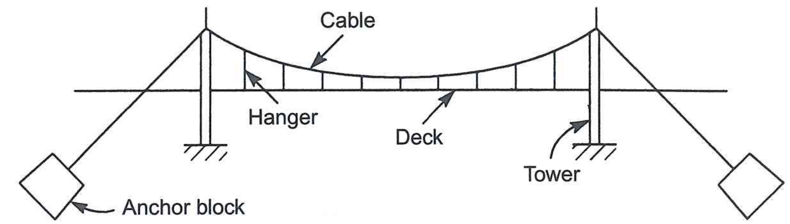


FIGURE 1.7

Suspension bridge.

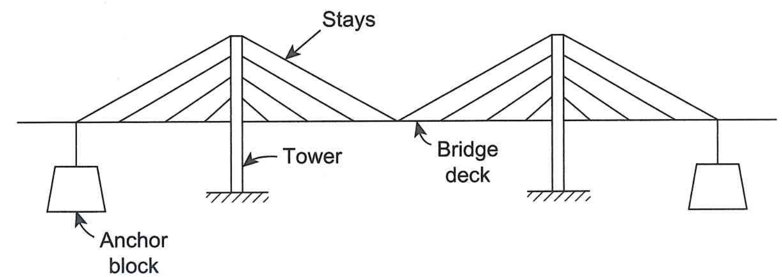


FIGURE 1.8

Cable-stayed bridge.

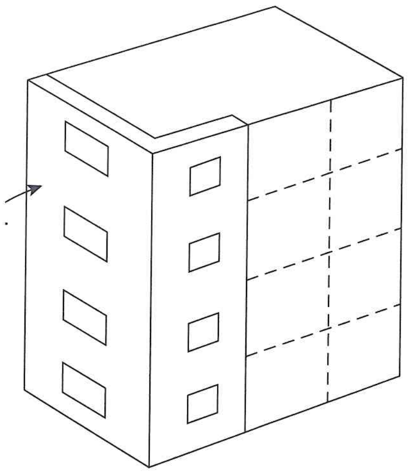
within the ground by massive anchor blocks. The cables carry hangers from which the bridge deck is suspended; a typical arrangement is shown in Fig. 1.7.

A weakness of suspension bridges is that, unless carefully designed, the deck is very flexible and can suffer large twisting displacements. A well-known example of this was the Tacoma Narrows suspension bridge in the US in which twisting oscillations were triggered by a wind speed of only 19 m/s. The oscillations increased in amplitude until the bridge collapsed approximately 1 h after the oscillations had begun. To counteract this tendency bridge decks are stiffened. For example, the Forth Road Bridge has its deck stiffened by a space truss while the later Severn Bridge uses an aerodynamic, torsionally stiff, tubular cross-section bridge deck.

An alternative method of supporting a bridge deck of moderate span is the cable-stayed system shown in Fig. 1.8. *Cable-stayed bridges* were developed in Germany after World War II when materials were in short supply and a large number of highway bridges, destroyed by military action, had to be rebuilt. The tension in the stays is maintained by attaching the outer ones to anchor blocks embedded in the ground. The stays can be a single system from towers positioned along the centre of the bridge deck or a double system where the cables are supported by twin sets of towers on both sides of the bridge deck.

Shear and core walls

Sometimes, particularly in high rise buildings, *shear* or *core walls* are used to resist the horizontal loads



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wall construction.

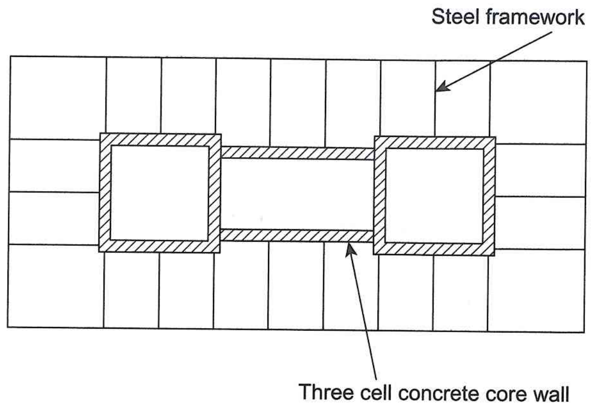


FIGURE 1.10

Sectional plan of core wall and steel structure.

ion parallel to its shortest horizontal dimension by a shear wall which would normally be of rein-
l concrete.

ternatively a lift shaft or service duct is used as the main horizontal load carrying member; this is n as a core wall. An example of core wall construction in a tower block is shown in cross section ;. 1.10. The three cell concrete core supports a suspended steel framework and houses a number :illary services in the outer cells while the central cell contains stairs, lifts and a central landing or n this particular case the core wall not only resists horizontal wind loads but also vertical loads) its self-weight and the suspended steel framework.

shear or core wall may be analysed as a very large, vertical, cantilever beam (see Fig. 1.15). A :m can arise, however, if there are openings in the walls, say, of a core wall which there would be, ırse, if the core was a lift shaft. In such a situation a computer-based method of analysis would bly be used.

inuum structures

ɔles of these are folded plate roofs, shells, floor slabs, etc. An arch dam is a three-dimensional con- n structure as are domed roofs, aircraft fuselages and wings. Generally, continuum structures e computer-based methods of analysis.

Support systems

ɪads applied to a structure are transferred to its foundations by its supports. In practice supports e rather complicated in which case they are simplified, or *idealized*, into a form that is much easier lyse. For example, the support shown in Fig. 1.11(a) allows the beam to rotate but prevents transla- ɔth horizontally and vertically. For the purpose of analysis it is represented by the idealized form

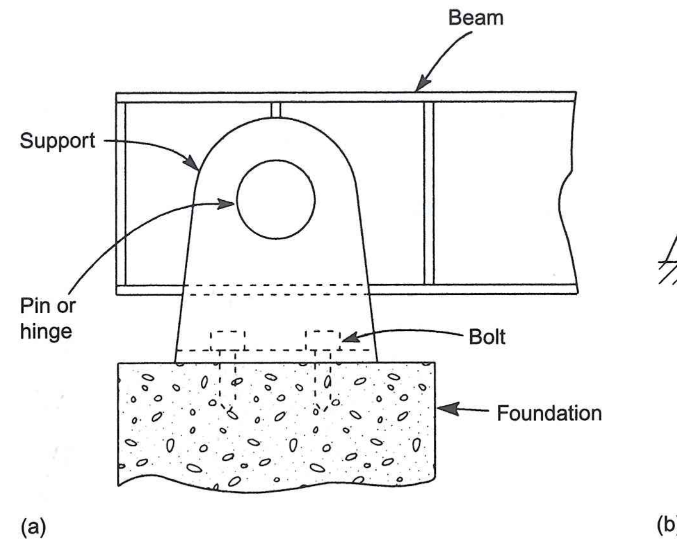


FIGURE 1.11

Idealization of a pinned support.

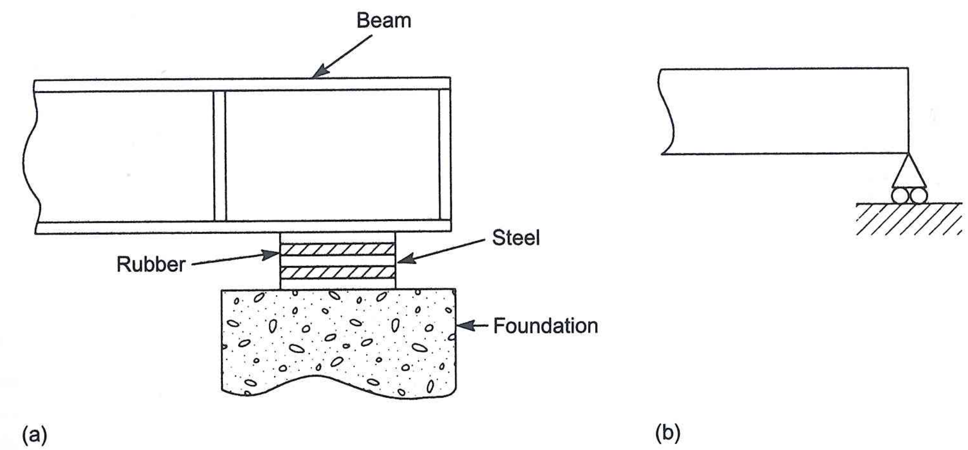


FIGURE 1.12

Idealization of a sliding or roller support.

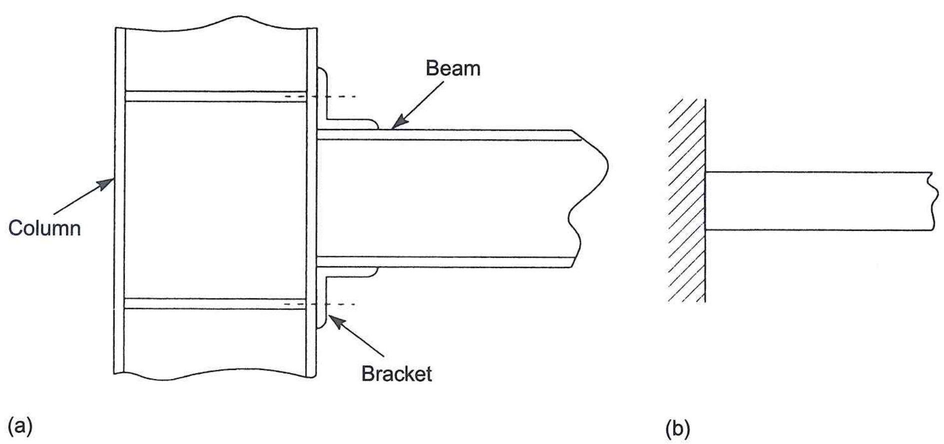
A beam that is supported at one end by a pinned support would not necessarily be supported in the same way at the other. One support of this type is sufficient to maintain the horizontal equilibrium of a beam and it may be advantageous to allow horizontal movement of the other end so that, for example, expansion and contraction caused by temperature variations do not cause additional stresses. Such a support may take the form of a composite steel and rubber bearing as shown in Fig. 1.12(a) or consist of a

g. 1.12(b) and is called a *roller support.* It is assumed that such a support allows horizontal movement ·otation but prevents movement vertically, up or down.

: is worth noting that a horizontal beam on two pinned supports would be statically indeterminate ther than purely vertical loads since, as we shall see in Section 2.5, there would be two vertical and horizontal components of support reaction but only three independent equations of statical ibrium.

ı some instances beams are supported in such a way that both translation and rotation are prevented. g. 1.13(a) the steel I-beam is connected through brackets to the flanges of a steel column and there- :annot rotate or move in any direction; the idealized form of this support is shown in Fig. 1.13(b) s called a *fixed, built-in* or *encastré support.* A beam that is supported by a pinned support and a roller ɔrt as shown in Fig. 1.14(a) is called a *simply supported beam;* note that the supports will not neces- be positioned at the ends of a beam. A beam supported by combinations of more than two pinned ·oller supports (Fig. 1.14(b)) is known as a *continuous beam.* A beam that is built-in at one end and ıt the other (Fig. 1.15(a)) is a *cantilever beam* while a beam that is built-in at both ends (Fig. 1.15 s a *fixed, built-in* or *encastré* beam.

7hen loads are applied to a structure, reactions are produced in the supports and in many :ural analysis problems the first step is to calculate their values. It is important, therefore, to



E 1.13

zation of a built-in support.

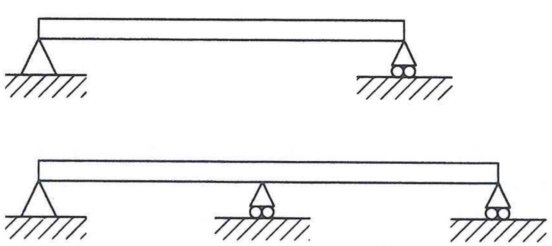


FIGURE 1.14

(a) Simply supported beam and (b)

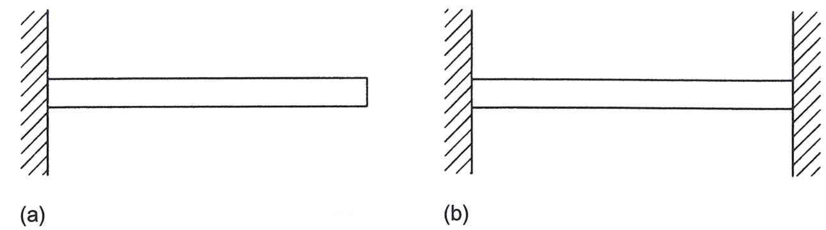


FIGURE 1.15

(a) Cantilever beam and (b) fixed or built-in beam.

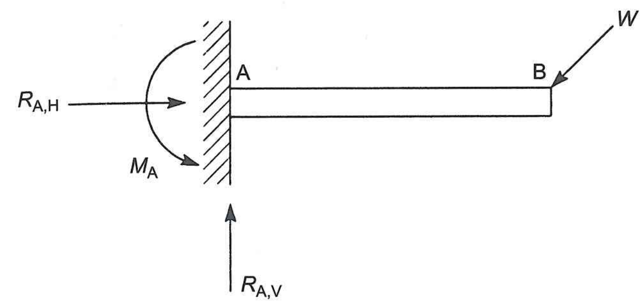


FIGURE 1.16

Support reactions in a cantilever beam subjected to an inclined load at its free end.

identify correctly the type of reaction associated with a particular support. Supports that prevent translation in a particular direction produce a force reaction in that direction while supports that prevent rotation cause moment reactions. For example, in the cantilever beam of Fig. 1.16, the applied load W has horizontal and vertical components which cause horizontal ($R_{A,H}$) and vertical ($R_{A,V}$) reactions of force at the built-in end A, while the rotational effect of W is balanced by the moment reaction M_A. We shall consider the calculation of support reactions in detail in Section 2.5.

1.5 Statically determinate and indeterminate structures

In many structural systems the principles of statical equilibrium (Section 2.4) may be used to determine support reactions and internal force distributions; such systems are called *statically determinate.* Systems for which the principles of statical equilibrium are insufficient to determine support reactions and/or internal force distributions, i.e. there are a greater number of unknowns than the number of equations of statical equilibrium, are known as *statically indeterminate* or *hyperstatic* systems. However, it is possible that even though the support reactions are statically determinate, the internal forces are not, and vice versa. For example, the truss in Fig. 1.17(a) is, as we shall see in Chapter 4, statically determinate both for support reactions and forces in the members whereas the truss shown in Fig. 1.17(b) is statically determinate only as far as the calculation of support reactions is concerned.

Another type of indeterminacy, *kinematic indeterminacy*, is associated with the ability to deform, or the degrees of freedom, of a structure and is discussed in detail in Section 16.3. A degree of freedom is a possible displacement of a joint (or node as it is often called) in a structure. For instance, a joint in a plane truss has three possible modes of displacement or degrees of freedom, two of translation in two

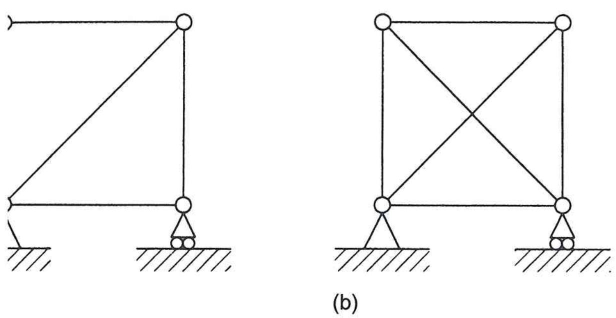


FIGURE 1.17
(a) Statically determinate truss and (b) statically indeterminate truss.

ıt in a three-dimensional space truss or frame possesses six degrees of freedom, three of translation ree mutually perpendicular directions and three of rotation about three mutually perpendicular

Analysis and design

: students in the early stages of their studies have only a vague idea of the difference between an tical problem and a design problem. We shall examine the various steps in the design procedure onsider the role of analysis in that procedure.

ıitially the structural designer is faced with a requirement for a structure to fulfil a particular role. may be a bridge of a specific span, a multistorey building of a given floor area, a retaining wall hav- required height and so on. At this stage the designer will decide on a possible form for the structure. xample, in the case of a bridge the designer must decide whether to use beams, trusses, arches or ; to support the bridge deck. To some extent, as we have seen, the choice is governed by the span red, although other factors may influence the decision. In Scotland, the Firth of Tay is crossed by a span bridge supported on columns, whereas the road bridge crossing the Firth of Forth is a suspen- ›ridge. In the latter case a large height clearance is required to accommodate shipping. In addition it sible that the designer may consider different schemes for the same requirement. Further decisions quired as to the materials to be used: steel, reinforced concrete, timber, etc.

aving decided on a particular system the loads on the structure are calculated. We have seen in ›n 1.2 that these comprise dead and live loads. Some of these loads, such as a floor load in an building, are specified in Codes of Practice while a particular Code gives details of how wind should be calculated. Of course the self-weight of the structure is calculated by the designer.

'hen the loads have been determined, the structure is *analysed*, i.e. the external and internal ; and moments are calculated, from which are obtained the internal stress distributions and he strains and displacements. The structure is then checked for *safety*, i.e. that it possesses suf- t strength to resist loads without danger of collapse, and for *serviceability*, which determines its / to carry loads without excessive deformation or local distress; Codes of Practice are used in ›rocedure. It is possible that this check may show that the structure is underdesigned (unsafe ı unserviceable) or overdesigned (uneconomic) so that adjustments must be made to the ʒement and/or the sizes of the members; the analysis and design check are then repeated.

ıalysis, as can be seen from the above discussion, forms only part of the complete design process ; concerned with a given structure subjected to given loads. Generally, there is a unique solution analytical problem whereas there may be one, two or more perfectly acceptable solutions to a

1.7 Structural and load idealization

Generally, structures are complex and must be *idealized* or simplified into a form that can be analysed. This idealization depends upon factors such as the degree of accuracy required from the analysis because, usually, the more sophisticated the method of analysis employed the more time consuming, and therefore the more costly, it is. A preliminary evaluation of two or more possible design solutions would not require the same degree of accuracy as the check on the finalized design. Other factors affecting the idealization include the type of load being applied, since it is possible that a structure will require different idealizations under different loads.

We have seen in Section 1.4 how actual supports are idealized. An example of structural idealization is shown in Fig. 1.18 where the simple roof truss of Fig. 1.18(a) is supported on columns and forms one of a series comprising a roof structure. The roof cladding is attached to the truss through purlins which connect each truss, and the truss members are connected to each other by gusset plates which may be riveted or welded to the members forming rigid joints. This structure possesses a high degree of statical indeterminacy and its analysis would probably require a computer-based approach. However, the assumption of a simple support system, the replacement of the rigid joints by pinned or hinged joints and the assumption that the forces in the members are purely axial, result, as we shall see in Chapter 4, in a statically determinate structure (Fig. 1.18(b)). Such an idealization might appear extreme but, so long as the loads are applied at the joints and the truss is supported at joints, the forces in the members are predominantly axial and bending moments and shear forces are negligibly small.

At the other extreme a continuum structure, such as a folded plate roof, would be idealized into a large number of *finite elements* connected at *nodes* and analysed using a computer; the *finite element method* is, in fact, an exclusively computer-based technique. A large range of elements is available in finite element packages including simple beam elements, plate elements, which can model both in-plane and out-of-plane effects, and three-dimensional 'brick' elements for the idealization of solid three-dimensional structures.

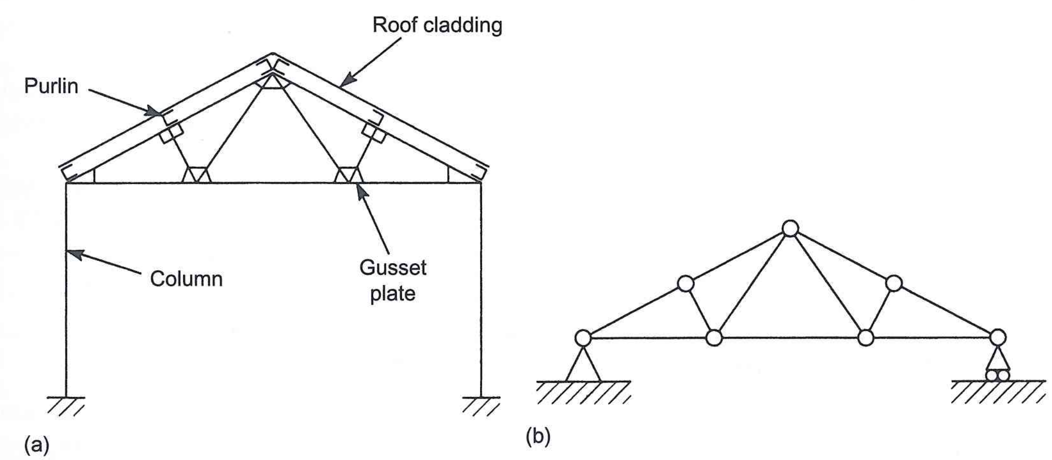
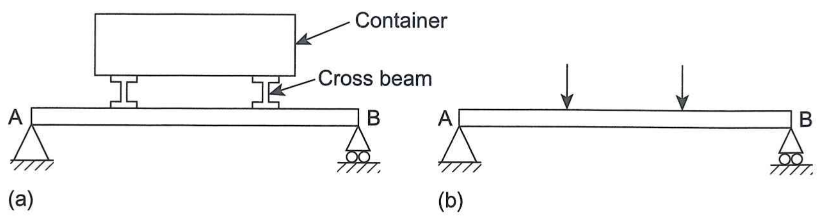
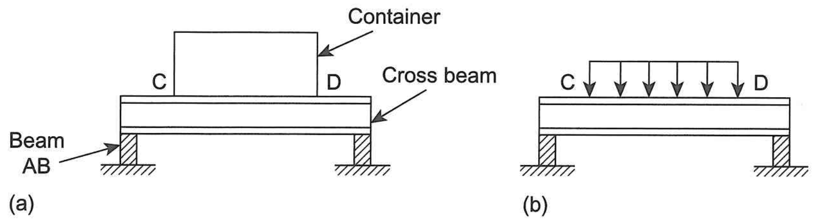


FIGURE 1.18



ꞮE 1.19

ization of a load system.



ꞮE 1.20

zation of a load system: uniformly distributed.

ı addition to the idealization of the structure, loads also, generally, need to be idealized. In 1.19(a) the beam AB supports two cross beams on which rests a container. There would, of course, second beam parallel to AB to support the other end of each cross beam. The flange of each cross ı applies a *distributed load* to the beam AB but if the flange width is small in relation to the span e beam they may be regarded as *concentrated loads* as shown in Fig. 1.19(b). In practice there is no thing as a concentrated load since, apart from the practical difficulties of applying one, a load act- ın zero area means that the stress (see Chapter 7) would be infinite and localized failure would :.

'he load carried by the cross beams, i.e. the container, would probably be applied along a consid- e portion of their length as shown in Fig. 1.20(a). In this case the load is said to be *uniformly buted* over the length CD of the cross beam and is represented as shown in Fig. 1.20(b).

ꞮDistributed loads need not necessarily be uniform but can be trapezoidal or, in more complicated be described by a mathematical function. Note that all the beams in Figs. 1.19 and 1.20 carry a rmly distributed load, their self-weight.

Structural elements

tures are made up of structural elements. For example, in frames these are beams and columns. cross sections of these structural elements vary in shape and depend on what is required in terms e forces to which they are subjected. Some common sections are shown in Fig. 1.21.

he solid square (or rectangular) and circular sections are not particularly efficient structurally.

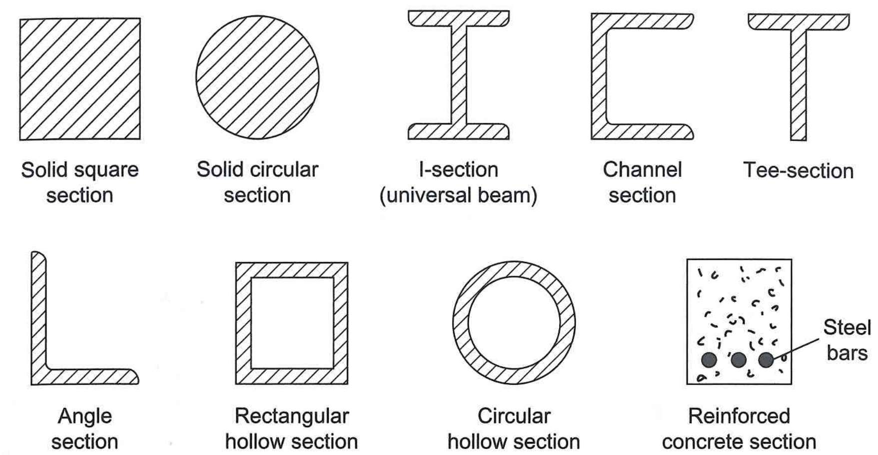


FIGURE 1.21

Structural elements.

(stretching forces acting along their length). In cases where the axial forces are compressive (shortening) then angle sections, channel sections, Tee-sections or I-sections would be preferred.

I-section and channel section beams are particularly efficient in carrying bending moments and shear forces (the latter are forces applied in the plane of a beam's cross section) as we shall see later.

The rectangular hollow (or square) section beam is also efficient in resisting bending and shear but is also used, as is the circular hollow section, as a column. A Universal Column has a similar cross section to that of the Universal Beam except that the flange width is greater in relation to the web depth.

Concrete, which is strong in compression but weak in tension, must be reinforced by steel bars on its tension side when subjected to bending moments. In many situations concrete beams are reinforced in both tension and compression zones and also carry shear force reinforcement.

Other types of structural element include box girder beams which are fabricated from steel plates to form tubular sections; the plates are stiffened along their length and across their width to prevent them buckling under compressive loads. Plate girders, once popular in railway bridge construction, have the same cross-sectional shape as a Universal Beam but are made up of stiffened plates and have a much greater depth than the largest standard Universal Beam. Reinforced concrete beams are sometimes cast integrally with floor slabs whereas in other situations a concrete floor slab may be attached to the flange of a Universal Beam to form a composite section. Timber beams are used as floor joists, roof trusses and, in laminated form, in arch construction and so on.

1.9 Materials of construction

A knowledge of the properties and behaviour of the materials used in structural engineering is essential if safe and long-lasting structures are to be built. Later we shall examine in some detail the properties of the more common construction materials but for the moment we shall review the materials available.

Steel

Steel is one of the most commonly used materials and is manufactured from iron ore which is first con-

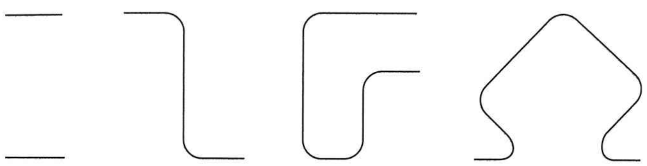


FIGURE 1.22

Examples of cold-formed sections.

·on, silicon, manganese, etc. added, the amounts depending on the particular steel being ıufactured.

Mild steel is the commonest type of steel and has a low carbon content. It is relatively strong, cheap ›roduce and is widely used for the sections shown in Fig. 1.21. It is a *ductile* material (see pter 8), is easily welded and because its composition is carefully controlled its properties are known ı reasonable accuracy. *High carbon steels* possess greater strength than mild steel but are less ductile reas *high yield steel* is stronger than mild steel but has a similar stiffness. High yield steel, as well as l steel, is used for reinforcing bars in concrete construction and very high strength steel is used for wires in prestressed concrete beams.

ow carbon steels possessing sufficient ductility to be bent cold are used in the manufacture of *cold-ıed* sections. In this process unheated thin steel strip passes through a series of rolls which gradually l it into the required section contour. Simple profiles, such as a channel section, may be produced s few as six stages whereas more complex sections may require 15 or more. Cold-formed sections ısed as lightweight roof purlins, stiffeners for the covers and sides of box beams and so on. Some cal sections are shown in Fig. 1.22.

Other special purpose steels are produced by adding different elements. For example, chromium is ed to produce stainless steel although this is too expensive for general structural use.

ıcrete

crete is produced by mixing cement, the commonest type being *ordinary Portland cement*, fine egate (sand), coarse aggregate (gravel, chippings) with water. A typical mix would have the ratio of ent/sand/coarse aggregate to be 1 : 2 : 4 but this can be varied depending on the required strength.

The tensile strength of concrete is roughly only 10% of its compressive strength and therefore, as ıave already noted, requires reinforcing in its weak tension zones and sometimes in its compression s.

ıber

ber falls into two categories, *hardwoods* and *softwoods*. Included in hardwoods are oak, beech, ash, ogany, teak, etc. while softwoods come from coniferous trees, such as spruce, pine and Douglas fir. dwoods generally possess a short grain and are not necessarily hard. For example, balsa is classed as rdwood because of its short grain but is very soft. On the other hand some of the long-grained soft-ds, such as pitch pine, are relatively hard.

Timber is a *naturally* produced material and its properties can vary widely due to varying quality significant defects. It has, though, been in use as a structural material for hundreds of years as a to any of the many cathedrals and churches built in the Middle Ages will confirm. Some of tim-disadvantages, such as warping and twisting, can be eliminated by using it in laminated form. *ood* is built up from several thin sheets glued together but with adjacent sheets having their grains ıing at 90° to each other. Large span roof arches are sometimes made in laminated form from tim-strips. Its susceptibility to the fungal attacks of wet and dry rot can be prevented by treatment as

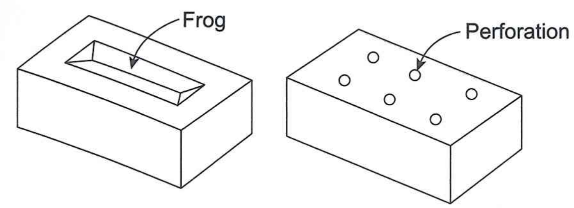


FIGURE 1.23

Types of brick.

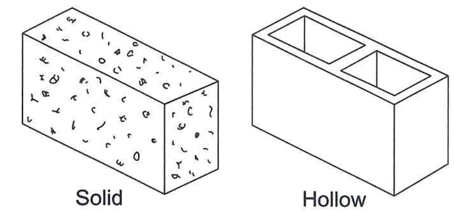


FIGURE 1.24

Concrete blocks.

Masonry

Masonry in structural engineering includes bricks, concrete blocks and stone. These are brittle materials, weak in tension, and are therefore used in situations where they are only subjected to compressive loads.

Bricks are made from clay shale which is ground up and mixed with water to form a stiff paste. This is pressed into moulds to form the individual bricks and then fired in a kiln until hard. An alternative to using individual moulds is the *extrusion process* in which the paste is squeezed through a rectangular-shaped die and then chopped into brick lengths before being fired.

Figure 1.23 shows two types of brick. One has indentations, called *frogs*, in its larger faces while the other, called a *perforated* brick, has holes passing completely through it; both these modifications assist the *bond* between the brick and the mortar and help to distribute the heat during the firing process. The holes in perforated bricks also allow a wall, for example, to be reinforced vertically by steel bars passing through the holes and into the foundations.

Engineering bricks are generally used as the main load bearing components in a masonry structure and have a minimum guaranteed crushing strength whereas *facing* bricks have a wide range of strengths but have, as the name implies, a better appearance. In a masonry structure the individual elements are the bricks while the complete structure, including the *mortar* between the joints, is known as *brickwork*.

Mortar commonly consists of a mixture of sand and cement the proportions of which can vary from 3:1 to 8:1 depending on the strength required; the lower the amount of sand the stronger the mortar. However, the strength of the mortar must not be greater than the strength of the masonry units otherwise cracking can occur.

Concrete blocks, can be solid or hollow as shown in Fig. 1.24, are cheap to produce and are made from special lightweight aggregates. They are rough in appearance when used for, say, insulation purposes and are usually covered by plaster for interiors or cement rendering for exteriors. Much finer facing blocks are also manufactured for exterior use and are not covered.

Stone, like timber, is a natural material and is, therefore, liable to have the same wide, and generally unpredictable, variation in its properties. It is expensive since it must be quarried, transported and then, if necessary, 'dressed' and cut to size. However, as with most natural materials, it can provide very attractive structures.

Aluminium

Pure aluminium is obtained from bauxite, is relatively expensive to produce, and is too soft and weak to act as a structural material. To overcome its low strength it is alloyed with elements such as magnetively high strength/low weight ratio is a marked advantage; aluminium is also a ductile material. In ıctural engineering aluminium sections are used for fabricating lightweight roof structures, window nes, etc. It can be extruded into complicated sections but the sections are generally smaller in size n the range available in steel.

st iron, wrought iron

ese are no longer used in modern construction although many old, existing structures contain them. t iron is a brittle material, strong in compression but weak in tension and contains a number of ourities which have a significant effect on its properties.

Wrought iron has a much less carbon content than cast iron, is more ductile but possesses a rela-ly low strength.

mposite materials

ne use is now being made of fibre reinforced polymers or *composites* as they are called. These are tweight, high strength materials and have been used for a number of years in the aircraft, automo- and boat building industries. They are, however, expensive to produce and their properties are not y understood.

Strong fibres, such as glass or carbon, are set in a matrix of plastic or epoxy resin which is then hanically and chemically protective. The fibres may be continuous or discontinuous and are gener- arranged so that their directions match those of the major loads. In sheet form two or more layers sandwiched together to form a *lay-up.*

In the early days of composite materials glass fibres were used in a plastic matrix, this is known as s reinforced plastic (GRP). More modern composites are carbon fibre reinforced plastics (CFRP). ıer composites use boron and Kevlar fibres for reinforcement.

Structural sections, as opposed to sheets, are manufactured using the *pultrusion* process in which es are pulled through a bath of resin and then through a heated die which causes the resin to den; the sections, like those of aluminium alloy, are small compared to the range of standard steel ions available.

10 The use of computers

nodern-day design offices most of the structural analyses are carried out using computer programs. ide variety of packages is available and range from relatively simple plane frame (two-dimensional) grams to more complex *finite element* programs which are used in the analysis of continuum struc-s. The algorithms on which these programs are based are derived from fundamental structural the- written in matrix form so that they are amenable to computer-based solutions. However, rather ı simply supplying data to the computer, structural engineers should have an understanding of the damental theory for without this basic knowledge it would be impossible for them to make an ssment of the limitations of the particular program being used. Unfortunately there is a tendency, icularly amongst students, to believe without question results in a computer printout. Only with an erstanding of how structures behave can the validity of these results be mentally checked.

The first few chapters of this book, therefore, concentrate on basic structural theory although, where ropriate, computer-based applications will be discussed. In later chapters computer methods, i.e. rix and finite element methods, are presented in detail.

CHAPTER 2

Principles of Statics

Statics, as the name implies, is concerned with the study of bodies at rest or, in other words, in equilibrium, under the action of a force system. Actually, a moving body is in equilibrium if the forces acting on it are producing neither acceleration nor deceleration. However, in structural engineering, structural members are generally at rest and therefore in a state of *statical equilibrium.*

In this chapter we shall discuss those principles of statics that are essential to structural and stress analysis; an elementary knowledge of vectors is assumed.

2.1 Force

The definition of a force is derived from Newton's First Law of Motion which states that a body will remain in its state of rest or in its state of uniform motion in a straight line unless compelled by an external force to change that state. Force is therefore associated with a *change* in motion, i.e. it causes acceleration or deceleration.

The basic unit of force in structural and stress analysis is the *Newton* (N) which is roughly a tenth of the weight of this book. This is a rather small unit for most of the loads in structural engineering so a more convenient unit, the *kilonewton* (kN) is often used.

$$1 \text{ kN} = 1000 \text{ N}$$

All bodies possess *mass* which is usually measured in *kilograms* (kg). The mass of a body is a measure of the quantity of matter in the body and, for a particular body, is invariable. This means that a steel beam, for example, having a given *weight* (the force due to gravity) on earth would weigh approximately six times less on the moon although its mass would be exactly the same.

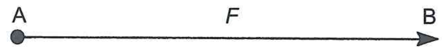
We have seen that force is associated with acceleration and Newton's Second Law of Motion tells us that

$$\text{force} = \text{mass} \times \text{acceleration}$$

Gravity, which is the pull of the earth on a body, is measured by the acceleration it imparts when a body falls; this is taken as 9.81 m/s^2 and is given the symbol g. It follows that the force exerted by gravity on a mass of 1 kg is

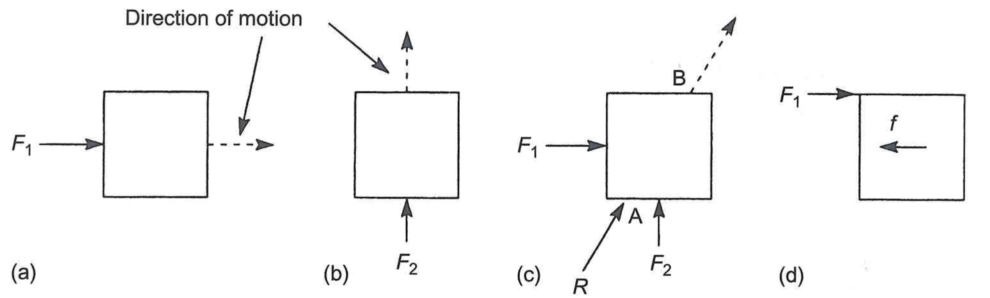
$$\text{force} = 1 \times 9.81$$

The Newton is defined as the force required to produce an acceleration of 1 m/s^2 in a mass of 1 kg which means that it would require a force of 9.81 N to produce an acceleration of 9.81 m/s^2 in a mass of 1 kg, i.e. the gravitational force exerted by a mass of 1 kg is 9.81 N. Frequently, in everyday usage, mass is taken to mean the weight of a body in kg.



URE 2.1

ɔresentation of a force by a vector.



JRE 2.2

on of forces on a cube.

We all have direct experience of force systems. The force of the earth's gravitational pull acts vertically ɾnwards on our bodies giving us weight; wind forces, which can vary in magnitude, tend to push us hori- tally. Therefore forces possess magnitude and direction. At the same time the effect of a force depends ɪn its position. For example, a door may be opened or closed by pushing horizontally at its free edge, but ıe same force is applied at any point on the vertical line through its hinges the door will neither open nor e. We see then that a force is described by its magnitude, direction and position and is therefore a *vector* ntity. As such it must obey the laws of vector addition, which is a fundamental concept that may be veri- experimentally.

Since a force is a vector it may be represented graphically as shown in Fig. 2.1, where the force F is sidered to be acting on an infinitesimally small particle at the point A and in a direction from left to ıt. The magnitude of F is represented, to a suitable scale, by the length of the line AB and its direction he direction of the arrow. In vector notation the force F is written as **F**.

Suppose a cube of material, placed on a horizontal surface, is acted upon by a force F_1 as shown in ı in Fig. 2.2(a). If F_1 is greater than the frictional force between the surface and the cube, the cube move in the direction of F_1. Again if a force F_2 is applied as shown in Fig. 2.2(b) the cube will ɾe in the direction of F_2. It follows that if F_1 and F_2 were applied simultaneously, the cube would ɾe in some inclined direction as though it were acted on by a single inclined force R (Fig. 2.2(c)); R ılled the *resultant* of F_1 and F_2.

Note that F_1 and F_2 (and R) are in a horizontal plane and that their lines of action pass through centre of gravity of the cube, otherwise rotation as well as translation would occur since, if F_1, were applied at one corner of the cube as shown in Fig. 2.2(d), the frictional force f, which may taken as acting at the center of the bottom face of the cube would, with F_1, form a couple Section 2.2).

The effect of the force R on the cube would be the same whether it was applied at the point A or at point B (so long as the cube is rigid). Thus a force may be considered to be applied at any point on ine of action, a principle known as the *transmissibility of a force*.

EXAMPLE 2.1

State the direction of motion of the block of material shown *in plan* in Fig. 2.3 (a)–(e) when it is subjected to the applied force, F, and is supported on a horizontal surface. The frictional force between the surface and the underside of the block is f.

(a) The block moves in the direction DB with no rotation.
(b) The block does not move translationally, possible rotation.
(c) The block moves parallel to AB and rotates in an anticlockwise sense.
(d) The block moves in a direction parallel to DA.
(e) The block moves in a direction parallel to AB with a clockwise rotation.
(f) The block moves in a direction parallel to DB with an anticlockwise rotation.

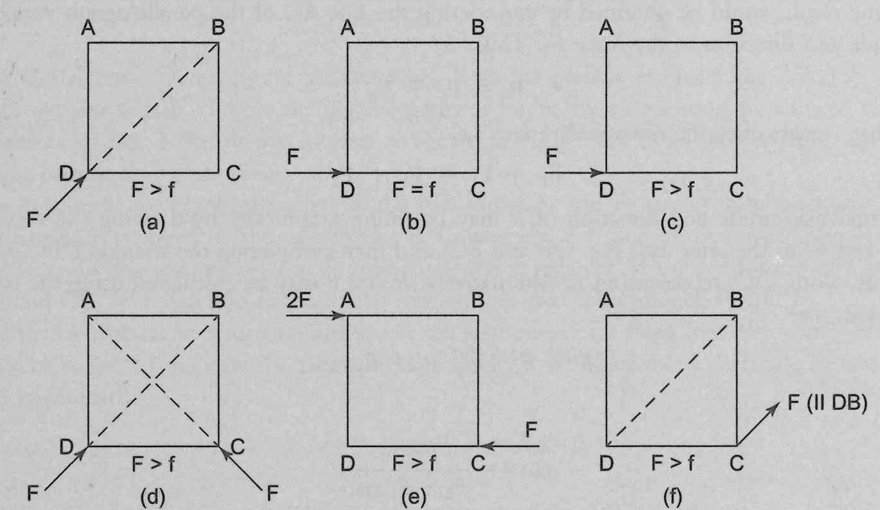


FIGURE 2.3

Force systems on block of Ex. 2.1

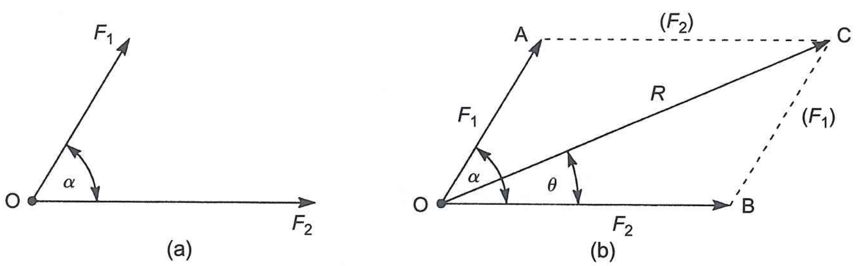
Parallelogram of forces

The resultant of two concurrent and coplanar forces, whose lines of action pass through a single point and lie in the same plane (Fig. 2.4(a)), may be found using the theorem of the parallelogram of forces which states that:

> *If two forces acting at a point are represented by two adjacent sides of a parallelogram drawn from that point their resultant is represented in magnitude and direction by the diagonal of the parallelogram drawn through the point.*

Thus in Fig. 2.4(b)R is the resultant of F_1 and F_2. This result may be verified experimentally or, alternatively, demonstrated to be true using the laws of vector addition. In Fig. 2.4(b) the side BC of the parallelogram is equal in magnitude and direction to the force F_1 represented by the side OA. Therefore, in vector notation

$$\mathbf{R} = \mathbf{F}_2 + \mathbf{F}_1$$



URE 2.4

sultant of two concurrent forces.

The same result would be obtained by considering the side AC of the parallelogram which is equal magnitude and direction to the force F_2. Thus

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$$

Note that vectors obey the *commutative law*, i.e.

$$\mathbf{F}_2 + \mathbf{F}_1 = \mathbf{F}_1 + \mathbf{F}_2$$

The actual magnitude and direction of R may be found graphically by drawing the vectors repre-ıting F_1 and F_2 to the *same scale* (i.e. OB and BC) and then completing the triangle OBC by drawing the vector, along OC, representing R. Alternatively, R and θ may be calculated using the trigonome- of triangles, i.e.

$$R^2 = F_1^2 + F_2^2 + 2F_1F_2 \cos\alpha \tag{2.1}$$

ɖ

$$\tan\theta = \frac{F_1 \sin\alpha}{F_2 + F_1 \cos\alpha} \tag{2.2}$$

XAMPLE 2.2

ɑlculate the magnitude and direction of the resultant of the two forces shown in Fig. 2.5; verify ɔur answer graphically.

From Eq. (2.1)

$$R^2 = 10^2 + 15^2 + 2 \times 10 \times 15 \cos 45^\circ$$

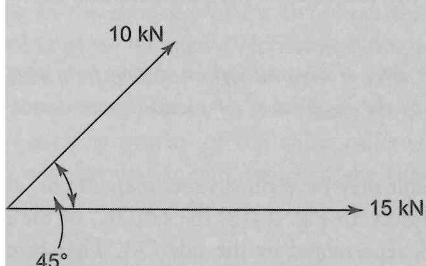


FIGURE 2.5

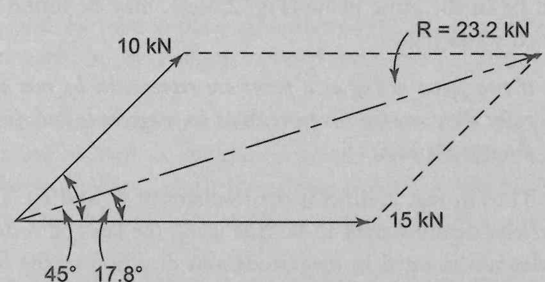


FIGURE 2.6

from which

$$R = 23.2 + \text{kN}$$

From Eq. (2.2)

$$\tan\theta = \frac{10 \sin 45^\circ}{15 + 10 \cos 45^\circ}$$

where θ is the inclination of R to the direction of the 15 kN force. Then

$$\theta = 17.8^\circ$$

The graphical solution is shown in Fig. 2.6.

In Fig. 2.4(a) both F_1 and F_2 are 'pulling away' from the particle at O. In Fig. 2.7(a) F_1 is a 'thrust' whereas F_2 remains a 'pull'. To use the parallelogram of forces the system must be reduced to either two 'pulls' as shown in Fig. 2.7(b) or two 'thrusts' as shown in Fig. 2.7(c). In all three systems we see that the effect on the particle at O is the same.

As we have seen, the combined effect of the two forces F_1 and F_2 acting simultaneously is the same as if they had been replaced by the single force R. Conversely, if R were to be replaced by F_1 and F_2 the effect would again be the same. F_1 and F_2 may therefore be regarded as the ***components*** of R in the directions OA and OB; R is then said to have been ***resolved*** into two components, F_1 and F_2.

Of particular interest in structural analysis is the resolution of a force into two components at right angles to each other. In this case the parallelogram of Fig. 2.4(b) becomes a rectangle in which $\alpha = 90^\circ$ (Fig. 2.8) and, clearly

$$F_2 = R\cos\theta \quad F_1 = R\sin\theta \tag{2.3}$$

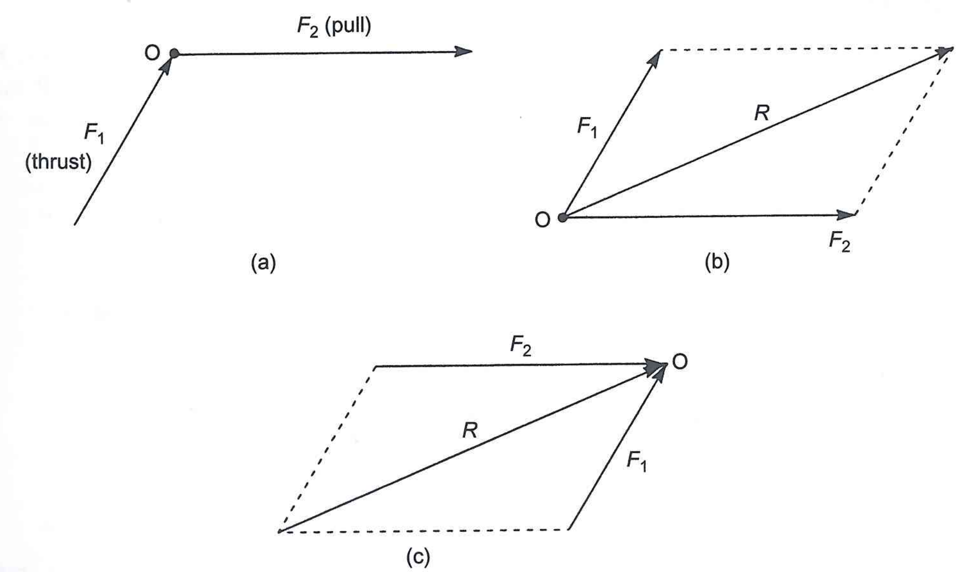


FIGURE 2.7

It follows from Fig. 2.8, or from Eqs (2.1) and (2.2), that

$$R^2 = F_1^2 + F_2^2 \qquad \tan\theta = \frac{F_1}{F_2} \tag{2.4}$$

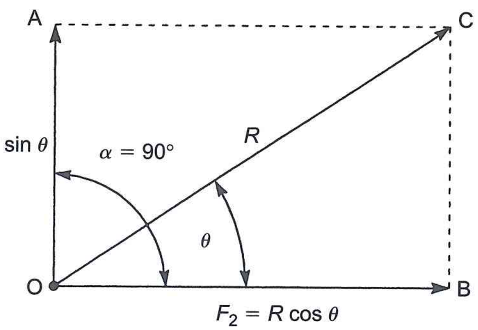
We note, by reference to Fig. 2.2(a) and (b), that a force does not induce motion in a direction per-
ɹdicular to its line of action; in other words a force has no effect in a direction perpendicular to itself.
.is may also be seen by setting $\theta = 90°$ in Eq. (2.3), then

$$F_1 = R \quad F_2 = 0$$

ɨ the component of R in a direction perpendicular to its line of action is zero.

e resultant of a system of concurrent forces

far we have considered the resultant of just two concurrent forces. The method used for that case
y be extended to determine the resultant of a system of any number of concurrent coplanar forces
:h as that shown in Fig. 2.9(a). Thus in the vector diagram of Fig. 2.9(b)



JRE 2.8

;olution of a force into two components at right angles.

$$\mathbf{R}_{12} = \mathbf{F}_1 + \mathbf{F}_2$$

where $\mathbf{R}_{12}$ is the resultant of $\mathbf{F}_1$ and $\mathbf{F}_2$. Further

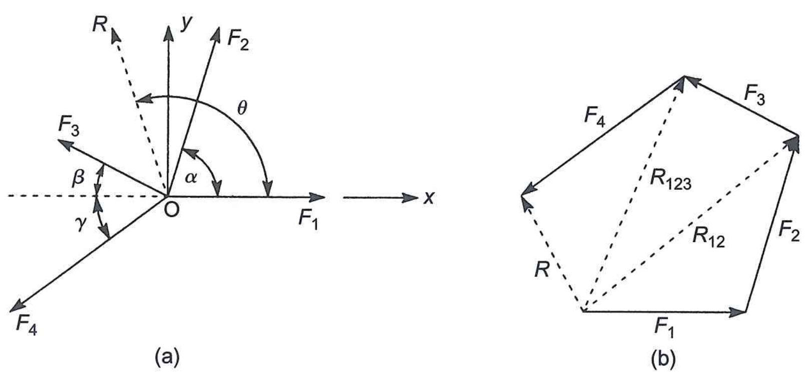
$$\mathbf{R}_{123} = \mathbf{R}_{12} + \mathbf{F}_3 = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

so that $\mathbf{R}_{123}$ is the resultant of $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$. Finally

$$\mathbf{R} = \mathbf{R}_{123} + \mathbf{F}_4 = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4$$

where $\mathbf{R}$ is the resultant of $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$.

The actual value and direction of $\mathbf{R}$ may be found graphically by constructing the vector



JRE 2.9

diagram of Fig. 2.9(b) to scale or by resolving each force into components parallel to two directions at right angles, say the x and y directions shown in Fig. 2.9(a). Then

$$F_x = F_1 + F_2\cos\alpha - F_3\cos\beta - F_4\cos\gamma$$
$$F_y = F_2\sin\alpha + F_3\sin\beta - F_4\sin\gamma$$

Then

$$R = \sqrt{F_x^2 + F_y^2}$$

and

$$\tan\theta = \frac{F_y}{F_x}$$

The forces F_1, F_2, F_3 and F_4 in Fig. 2.9(a) do not have to be taken in any particular order when constructing the vector diagram of Fig. 2.9(b). Identical results for the magnitude and direction of R are obtained if the forces in the vector diagram are taken in the order F_1, F_4, F_3, F_2 as shown in Fig. 2.10 or, in fact, are taken in any order so long as the directions of the forces are adhered to and one force vector is drawn from the end of the previous force vector.

Equilibrant of a system of concurrent forces

In Fig. 2.4(b) the resultant R of the forces F_1 and F_2 represents the combined effect of F_1 and F_2 on the particle at O. It follows that this effect may be eliminated by introducing a force R_E which is equal in magnitude but opposite in direction to R at O, as shown in Fig. 2.11(a). R_E is known at the *equilibrant* of F_1 and F_2 and the particle at O will then be in *equilibrium* and remain stationary. In other words the forces F_1, F_2 and R_E are in equilibrium and, by reference to Fig. 2.4(b), we see that these three forces may be represented by the triangle of vectors OBC as shown in Fig. 2.11(b). This result leads directly to the law of the *triangle of forces* which states that:

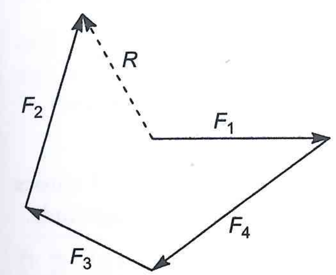


FIGURE 2.10

Alternative construction of force diagram for system of Fig. 2.9(a).

> *If three forces acting at a point are in equilibrium they may be represented in magnitude and direction by the sides of a triangle taken in order.*

The law of the triangle of forces may be used in the analysis of a plane, pin-jointed truss in which, say, one of three concurrent forces

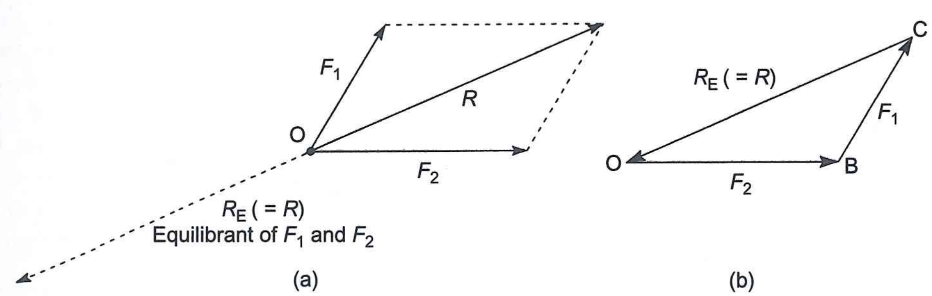
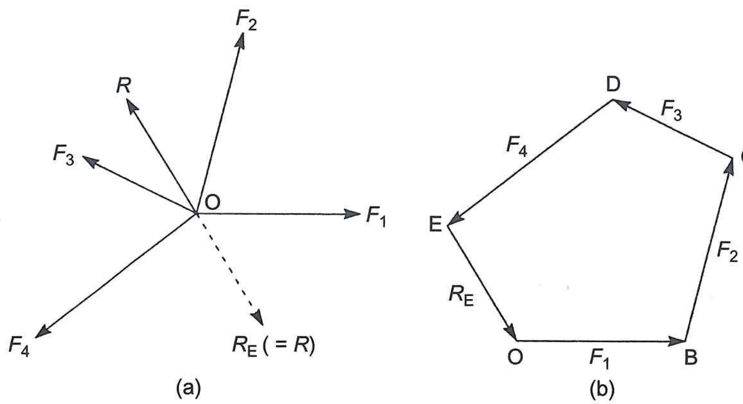


FIGURE 2.11



URE 2.12

ıilibrant of a number of concurrent forces.

ɪnown in magnitude and direction but only the lines of action of the other two. The law enables us find the magnitudes of the other two forces and also the direction of their lines of action.

The above arguments may be extended to a system comprising any number of concurrent forces. In force system of Fig. 2.9(a), R_E, shown in Fig. 2.12(a), is the equilibrant of the forces F_1, F_2, F_3 and Then F_1, F_2, F_3, F_4 and R_E may be represented by the force polygon OBCDE as shown in . 2.12(b).

The law of the *polygon of forces* follows:

If a number of forces acting at a point are in equilibrium they may be represented in magnitude and direction by the sides of a closed polygon taken in order.

Again, the law of the polygon of forces may be used in the analysis of plane, pin-jointed trusses ere several members meet at a joint but where no more than two forces are unknown in magnitude.

e resultant of a system of non-concurrent forces

most structural problems the lines of action of the different forces acting on the structure do not et at a single point; such a force system is non-concurrent.

Consider the system of non-concurrent forces shown in Fig. 2.13(a); their resultant may be found phically using the parallelogram of forces as demonstrated in Fig. 2.13(b). Produce the lines ıction of F_1 and F_2 to their point of intersection, I_1. Measure $I_1A = F_1$ and $I_1B = F_2$ to the same e, then complete the parallelogram I_1ACB; the diagonal CI_1 represents the resultant, R_{12}, of F_1 l F_2. Now produce the line of action of R_{12} backwards to intersect the line of action of F_3 at I_2. asure $I_2D = R_{12}$ and $I_2F = F_3$ to the same scale as before, then complete the parallelogram I_2DEF; diagonal $I_2E = R_{123}$, the resultant of R_{12} and F_3. It follows that $R_{123} = R$, the resultant of F_1, F_2 l F_3. Note that only the line of action and the magnitude of R can be found, not its point of action, ɔe the vectors F_1, F_2 and F_3 in Fig. 2.13(a) define the lines of action of the forces, not their points ıction.

If the points of action of the forces are known, defined, say, by coordinates referred to a ɪvenient xy axis system, the magnitude, direction and point of action of their resultant may be

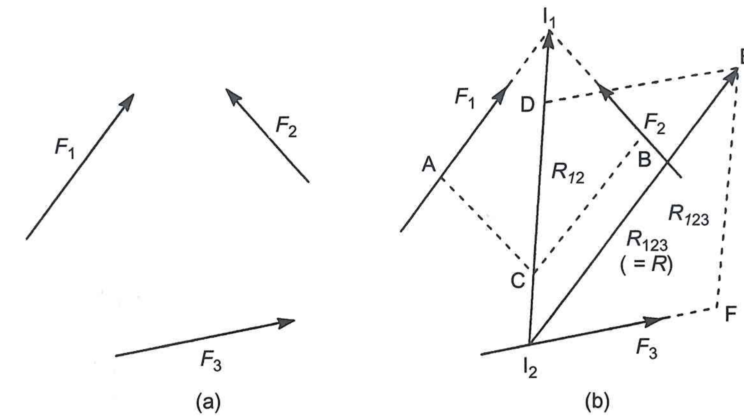


FIGURE 2.13

Resultant of a system of non-concurrent forces.

magnitude and position of the resultants R_x and R_y of each set of components using the method described in Section 2.3 for a system of parallel forces. The resultant R of the force system is then given by

$$R = \sqrt{R_x^2 + R_y^2}$$

and its point of action is the point of intersection of R_x and R_y; finally, its inclination θ to the x axis, say, is

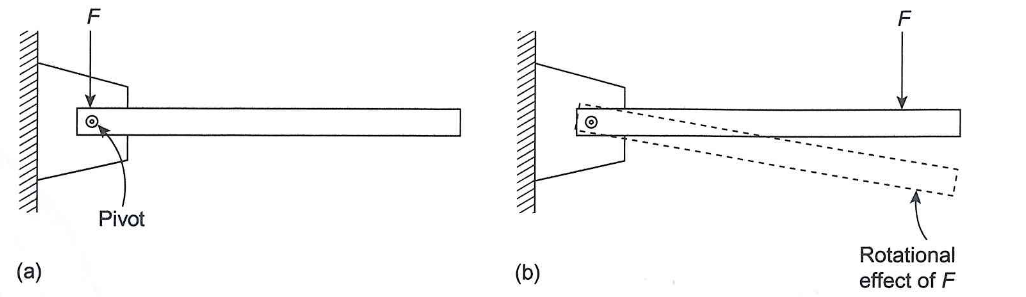
$$\theta = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

2.2 Moment of a force

So far we have been concerned with the translational effect of a force, i.e. the tendency of a force to move a body in a straight line from one position to another. A force may, however, exert a rotational effect on a body so that the body tends to turn about some given point or axis.

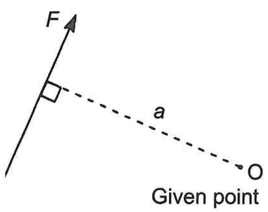
Figure 2.14(a) shows the cross section of, say, a door that is attached to a wall by a pivot and bracket arrangement which allows it to rotate in a horizontal plane. A horizontal force, F, whose line of action passes through the pivot, will have no rotational effect on the door but when applied at some distance along the door (Fig. 2.14(b)) will cause it to rotate about the pivot. It is common experience that the nearer the pivot the force F is applied the greater must be its magnitude to cause rotation. At the same time its effect will be greatest when it is applied at right angles to the door.

In Fig. 2.14(b) F is said to exert a *moment* on the door about the pivot. Clearly the rotational effect of F depends upon its magnitude and also on its distance from the pivot. We therefore define the moment of a force, F, about a given point O (Fig. 2.15) as the product of the force and the perpendicular distance of its line of action from the point. Thus, in Fig. 2.15, the moment, M, of F about O is given by



URE 2.14

:ational effect of a force.



URE 2.15

ment of a force about a
en point.

where '*a*' is known as the *lever arm* or *moment arm* of F about O; note that the units of a moment are the units of force × distance.

It can be seen from the above that a moment possesses both magnitude and a rotational sense. For example, in Fig. 2.15, F exerts a clockwise moment about O. A moment is therefore a vector (an alternative argument is that the product of a vector, F, and a scalar, a, is a vector). It is conventional to represent a moment vector graphically by a double-headed arrow, where the direction of the arrow designates a clockwise moment when looking in the direction of the arrow. Therefore, in Fig. 2.15, the moment $M(= Fa)$ would be represented by a double-headed arrow through O with its direction into the plane of the paper.

Moments, being vectors, may be resolved into components in the same way as forces. Consider the ment, M (Fig. 2.16(a)), in a plane inclined at an angle θ to the xz plane. The component of M in xz plane, M_{xz}, may be imagined to be produced by rotating the plane containing M through the gle θ into the xz plane. Similarly, the component of M in the yz plane, M_{yz}, is obtained by rotating plane containing M through the angle $90 - \theta$. Vectorially, the situation is that shown in Fig. 2.16 , where the directions of the arrows represent clockwise moments when viewed in the directions of arrows. Then

$$M_{xz} = M\cos\theta \quad M_{yz} = M\sin\theta$$

The action of a moment on a structural member depends upon the plane in which it acts. : example, in Fig. 2.17(a), the moment, M, which is applied in the longitudinal vertical plane symmetry, will cause the beam to bend in a vertical plane. In Fig. 2.17(b) the moment, M, is applied he plane of the cross section of the beam and will therefore produce twisting; in this case M is called *rque.*

uples

nsider the two coplanar, equal and parallel forces F which act in opposite directions as shown Fig. 2.18. The sum of their moments, M_O, about *any* point O in their plane is

$$M_O = F \times BO - F \times AO$$

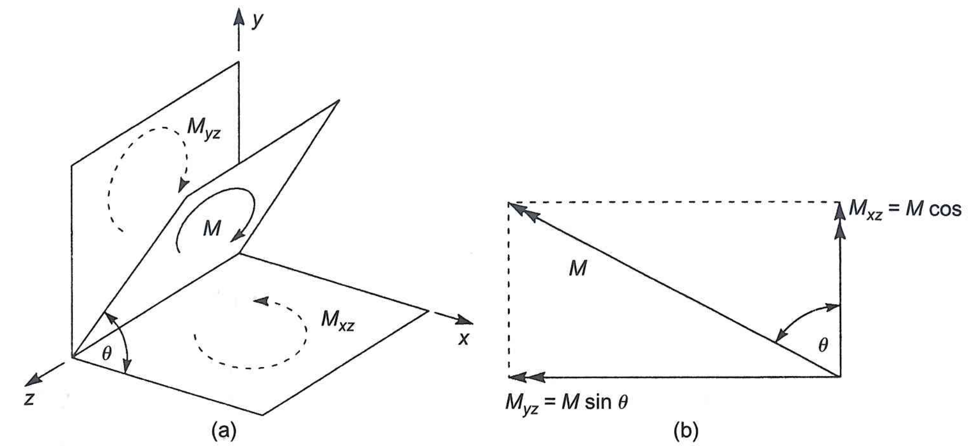


FIGURE 2.16

Resolution of a moment.

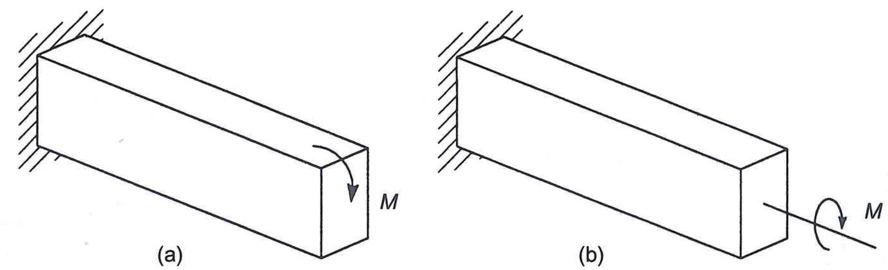


FIGURE 2.17

Action of a moment in different planes.

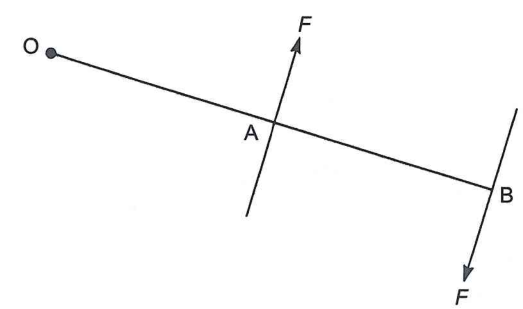
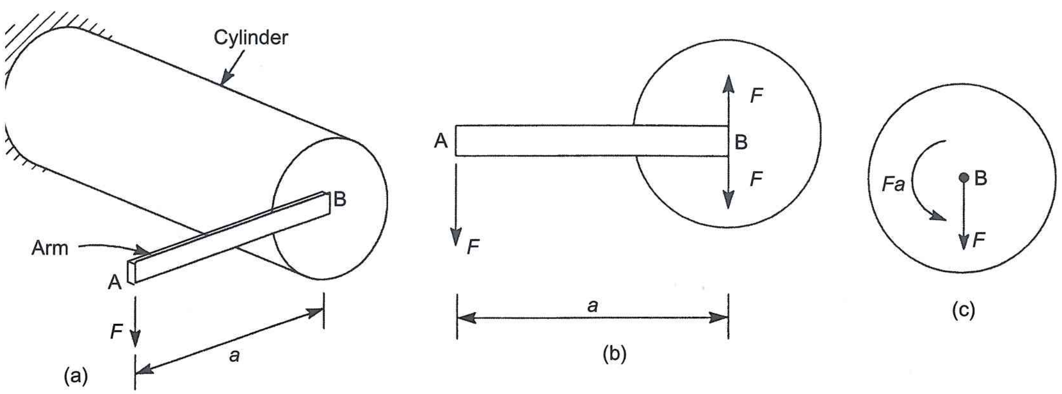


FIGURE 2.18



URE 2.19

ıivalent force system.

ere OAB is perpendicular to both forces. Then

$$Mo = F(BO - AO) = F \times AB$$

l we see that the sum of the moments of the two forces F about any point in their plane is equal to product of one of the forces and the perpendicular distance between their lines of action; this sys- ı is termed a *couple* and the distance AB is the *arm* or *lever arm* of the couple.

Since a couple is, in effect, a pure moment (not to be confused with the moment of a force about a cific point which varies with the position of the point) it may be resolved into components in the ıe way as the moment M in Fig. 2.16.

uivalent force systems

structural analysis it is often convenient to replace a force system acting at one point by an equiva- t force system acting at another. For example, in Fig. 2.19(a), the effect on the cylinder of the force cting at A on the arm AB may be determined as follows.

If we apply equal and opposite forces F at B as shown in Fig. 2.19(b), the overall effect on the ınder is unchanged. However, the force F at A and the equal and opposite force F at B form ouple which, as we have seen, has the same moment (Fa) about any point in its plane. Thus the ɜle force F at A may be replaced by a single force F at B together with a moment equal to Fa as wn in Fig. 2.19(c). The effects of the force F at B and the moment (actually a torque) Fa may be :ulated separately and then combined using the principle of superposition (see Section 3.7).

3 The resultant of a system of parallel forces

ce, as we have seen, a system of forces may be replaced by their resultant, it follows that a ticular action of a force system, say the combined moments of the forces about a point, must identical to the same action of their resultant. This principle may be used to determine the

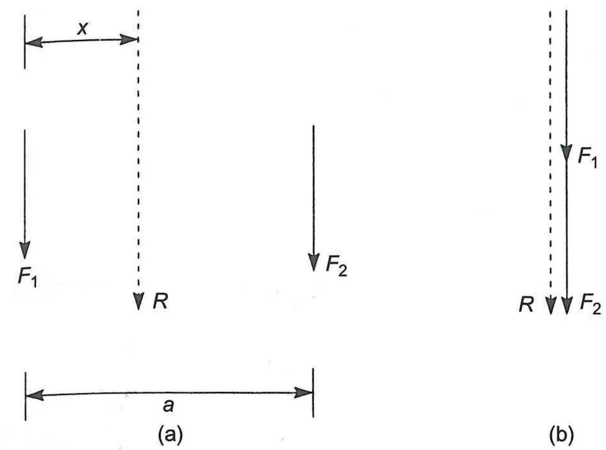


FIGURE 2.20

Resultant of a system of parallel forces.

The point of intersection of the lines of action of F_1 and F_2 is at infinity so that the parallelogram of forces (Fig. 2.4(b)) degenerates into a straight line as shown in Fig. 2.20(b) where, clearly

$$R = F_1 + F_2 \qquad (2.6)$$

The position of the line of action of R may be found using the principle stated above, i.e. the sum of the moments of F_1 and F_2 about any point must be equivalent to the moment of R about the same point. Thus from Fig. 2.20(a) and taking moments about, say, the line of action of F_1 we have

$$F_2 a = Rx = (F_1 + F_2)x$$

Hence

$$x = \frac{F_2}{F_1 + F_2} a \qquad (2.7)$$

Note that the action of R is *equivalent* to that of F_1 and F_2, so that, in this case, we equate clockwise to clockwise moments.

The principle of equivalence may be extended to any number of parallel forces irrespective of their directions and is of particular use in the calculation of the position of centroids of area, as we shall see in Section 9.6.

EXAMPLE 2.3

Find the magnitude and position of the line of action of the resultant of the force system shown in Fig. 2.21.

In this case the polygon of forces (Fig. 2.9(b)) degenerates into a straight line and

$$R = 2 - 3 + 6 + 1 = 6 \text{ kN} \qquad \text{(i)}$$

Suppose that the line of action of R is at a distance x from the 2 kN force, then, taking moments about the 2 kN force

$$Rx = -3 \times 0.6 + 6 \times 0.9 + 1 \times 1.2$$

Substituting for R from Eq. (i) we have

$$6x = -1.8 + 5.4 + 1.2$$

which gives

$$x = 0.8 \text{ m}$$

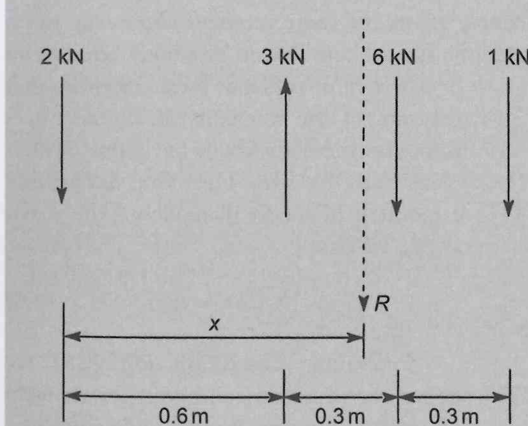


FIGURE 2.21

Force system of Ex. 2.3.

We could, in fact, take moments about any point, say now the 6 kN force. Then

$$R(0.9 - x) = 2 \times 0.9 - 3 \times 0.3 - 1 \times 0.3$$

so that

$$x = 0.8 \text{ m as before}$$

Note that in the second solution, anticlockwise moments have been selected as positive.

.4 Equilibrium of force systems

'e have seen in Section 2.1 that, for a particle or a body to remain stationary, i.e. in statical equilib- ım, the resultant force on the particle or body must be zero. It follows that if a body (generally in ructural analysis we are concerned with bodies, i.e. structural members, not particles) is not to move a particular direction, the resultant force in that direction must be zero. Furthermore, the prevention the movement of a body in two directions at right angles ensures that the body will not move in any rection at all. Then, for such a body to be in equilibrium, the sum of the components of all the forces ting on the body in any two mutually perpendicular directions must be zero. In mathematical terms ıd choosing, say, the x and y directions as the mutually perpendicular directions, the condition may written

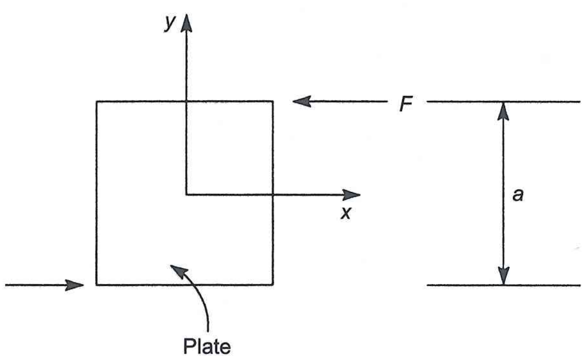
$$\sum F_x = 0 \quad \sum F_y = 0 \tag{2.8}$$

However, the condition specified by Eq. (2.8) is not sufficient to guarantee the equilibrium of body acted on by a system of coplanar forces. For example, in Fig. 2.22 the forces F acting on a plate sting on a horizontal surface satisfy the condition $\Sigma F_x = 0$ (there are no forces in the y direction so at $\Sigma F_y = 0$ is automatically satisfied), but form a couple Fa which will cause the plate to rotate in an ıticlockwise sense so long as its magnitude is sufficient to overcome the frictional resistance between e plate and the surface. We have also seen that a couple exerts the same moment about any point its plane so that we may deduce a further condition for the statical equilibrium of a body acted upon by a system of coplanar forces, namely, that the sum of the moments of all the forces acting on the body about *any* point in their plane must be zero. Therefore, designating a moment in the xy plane about the z axis as M_z, we have

$$\sum M_z = 0 \tag{2.9}$$

Combining Eqs (2.8) and (2.9) we obtain the necessary conditions for a system of coplanar forces to be in equilibrium, i.e.

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_z = 0 \tag{2.10}$$



:URE 2.22

The above arguments may be extended to a three-dimensional force system which is, again, referred to an xyz axis system. Thus for equilibrium

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0 \tag{2.11}$$

and

$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0 \tag{2.12}$$

2.5 Calculation of support reactions

The conditions of statical equilibrium, Eq. (2.10), are used to calculate reactions at supports in structures so long as the support system is statically determinate (see Section 1.5). Generally the calculation of support reactions is a necessary preliminary to the determination of internal force and stress distributions and displacements.

EXAMPLE 2.4

Calculate the support reactions in the simply supported beam ABCD shown in Fig. 2.23.

The different types of support have been discussed in Section 1.4. In Fig. 2.23 the support at A is a pinned support which allows rotation but no translation in any direction, while the support at D allows rotation and translation in a horizontal direction but not in a vertical direction. Therefore there will be no moment reactions at A or D and only a vertical reaction at D, R_D. It follows that the horizontal component of the 5 kN load can only be resisted by the support at A, $R_{A,H}$, which, in addition, will provide a vertical reaction, $R_{A,V}$.

Since the forces acting on the beam are coplanar, Eqs. (2.10) are used. From the first of these, i.e. $\sum F_x = 0$, we have

$$R_{A,H} - 5\cos 60° = 0$$

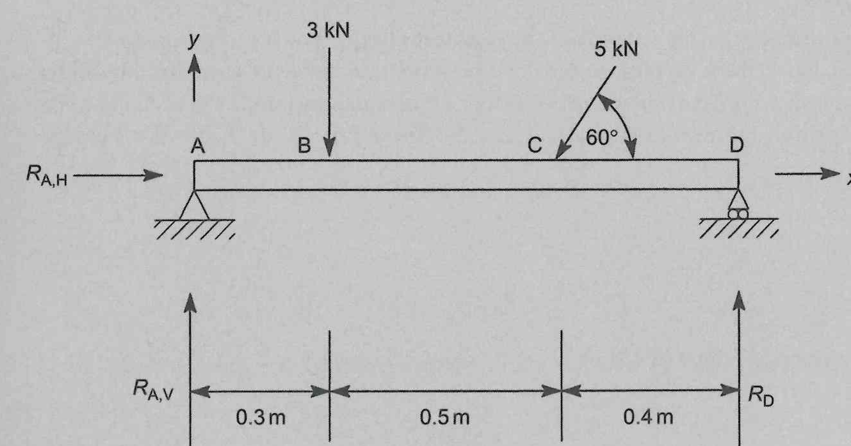


FIGURE 2.23

Beam of Ex. 2.4.

which gives

$$R_{A,H} = 2.5 \text{ kN}$$

The use of the second equation, $\Sigma F_y = 0$, at this stage would not lead directly to either $R_{A,V}$ or R_D since both would be included in the single equation. A better approach is to use the moment equation, $\Sigma M_z = 0$, and take moments about either A or D (it is immaterial which), thereby eliminating one of the vertical reactions. Taking moments, say, about D, we have

$$R_{A,V} \times 1.2 - 3 \times 0.9 - (5 \sin 60°) \times 0.4 = 0 \qquad \text{(i)}$$

Note that in Eq. (i) the moment of the 5 kN force about D may be obtained either by calculating the perpendicular distance of its line of action from D (0.4 sin 60°) or by resolving it into vertical and horizontal components (5 sin 60° and 5 cos 60°, respectively) where only the vertical component exerts a moment about D. From Eq. (i)

$$R_{A,V} = 3.7 \text{ kN}$$

The vertical reaction at D may now be found using $\sum F_y = 0$ or by taking moments about A, which would be slightly lengthier. Thus

$$R_D + R_{A,V} - 3 - 5 \sin 60° = 0$$

so that

$$R_D = 3.6 \text{ kN}$$

EXAMPLE 2.5

Calculate the reactions at the support in the cantilever beam shown in Fig. 2.24.

The beam has a fixed support at A which prevents translation in any direction and also rotation. The loads applied to the beam will therefore induce a horizontal reaction, $R_{A,H}$, at A and a vertical reaction, $R_{A,V}$, together with a moment reaction M_A. Using the first of Eqs. (2.10), $\Sigma F_x = 0$, we obtain

$$R_{A,H} - 2 \cos 45° = 0$$

whence

$$R_{A,H} = 1.4 \text{ kN}$$

From the second of Eqs. (2.10), $\sum F_y = 0$

$$R_{A,V} - 5 - 2 \sin 45° = 0$$

which gives

$$R_{A,V} = 6.4 \text{ kN}$$

Finally from the third of Eqs. (2.10), $\Sigma M_z = 0$, and taking moments about A, thereby eliminating $R_{A,H}$ and $R_{A,V}$

$$M_A - 5 \times 0.4 - (2 \sin 45°) \times 1.0 = 0$$

from which

$$M_A = 3.4 \text{ kN m}$$

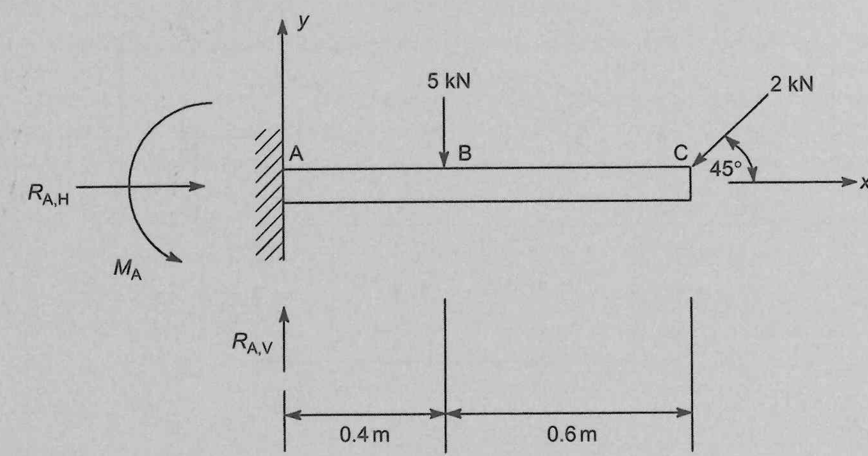


FIGURE 2.24

Beam of Ex. 2.5.

In Exs 2.4 and 2.5, the directions or sense of the support reactions is reasonably obvious. However, where this is not the case, a direction or sense is assumed which, if incorrect, will result in a negative value.

Occasionally the resultant reaction at a support is of interest. In Ex. 2.4 the resultant reaction at A is found using the first of Eqs. (2.4), i.e.

$$R_A^2 = R_{A,H}^2 + R_{A,V}^2$$

which gives

$$R_A^2 = 2.5^2 + 3.7^2$$

so that

$$R_A = 4.5 \text{ kN}$$

The inclination of R_A to, say, the vertical is found from the second of Eqs. (2.4). Thus

$$\tan\theta = \frac{R_{A,H}}{R_{A,V}} = \frac{2.5}{3.7} = 0.676$$

from which

$$\theta = 34.0°$$

EXAMPLE 2.6

Calculate the reactions at the supports in the plane truss shown in Fig. 2.25.

The truss is supported in the same manner as the beam in Ex. 2.4 so that there will be horizontal and vertical reactions at A and only a vertical reaction at B.

The angle of the truss, α, is given by

$$\alpha = \tan^{-1}\left(\frac{2.4}{3}\right) = 38.7^\circ$$

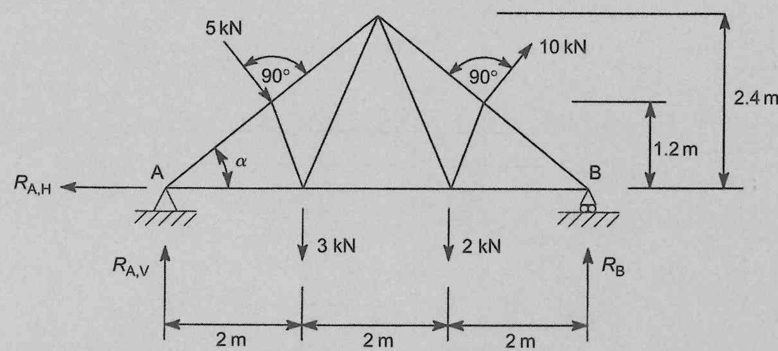


FIGURE 2.25

Truss of Ex. 2.6.

From the first of Eqs. (2.10) we have

$$R_{A,H} - 5\sin 38.7^\circ - 10\sin 38.7^\circ = 0$$

from which

$$R_{A,H} = 9.4\text{ kN}$$

Now taking moments about B, say, $(\sum M_B = 0)$

$$R_{A,V} \times 6 - (5\cos 38.7^\circ) \times 4.5 + (5\sin 38.7^\circ) \times 1.2 + (10\cos 38.7^\circ)$$
$$\times 1.5 + (10\sin 38.7) \times 1.2 - 3 \times 4 - 2 \times 2 = 0$$

which gives

$$R_{A,V} = 1.8\text{ kN}$$

Note that in the moment equation it is simpler to resolve the 5 kN and 10 kN loads into horizontal and vertical components at their points of application and then take moments rather than calculate the perpendicular distance of each of their lines of action from B.

The reaction at B, R_B, is now most easily found by resolving vertically $(\sum F_y = 0)$, i.e.

$$R_B + R_{A,B} - 5\cos 38.7^\circ + 10\cos 38.7^\circ - 3 - 2 = 0$$

which gives

$$R_B = -0.7\text{ kN}$$

In this case the negative sign of R_B indicates that the reaction is downward, not upward, as initially

PROBLEMS

P.2.1. State the direction of motion of the cube of material shown *in plan* in Fig. P.2.1(a)–(d) which is subjected to an applied force, *F*, and which is supported on a horizontal surface where the frictional force between the surface and the underside of the cube is *f*.

Ans.
(a) Translation parallel to BA.
(b) Translation parallel to BD, clockwise rotation.
(c) No translation, possible clockwise rotation.
(d) Translation at an angle of 28.7° to AD, anticlockwise rotation.

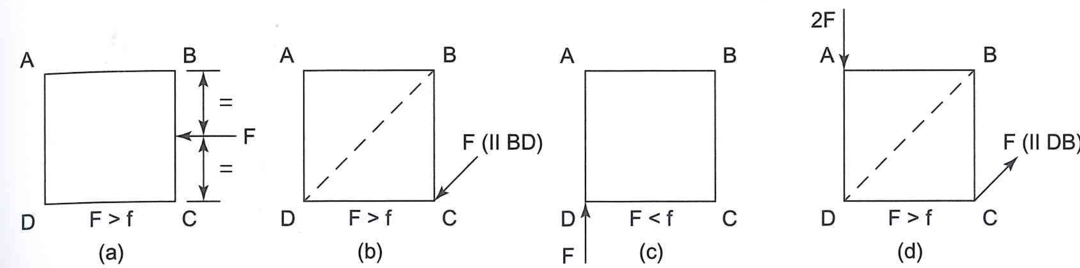


FIGURE P.2.1

P.2.2. Determine the magnitude and inclination of the resultant of the two forces acting at the point O in Fig. P.2.2 (a) by a graphical method and (b) by calculation.

Ans. 21.8 kN, 23.4° to the direction of the 15 kN load.

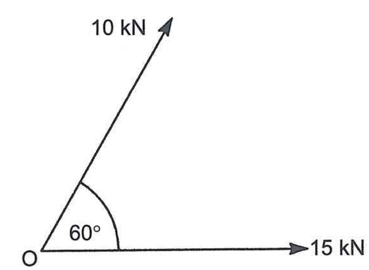
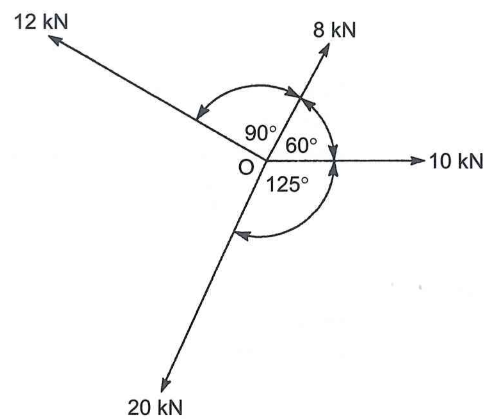


FIGURE P.2.2

P.2.3. Determine the magnitude and inclination of the resultant of the system of concurrent forces shown in Fig. P.2.3 (a) by a graphical method and (b) by calculation.

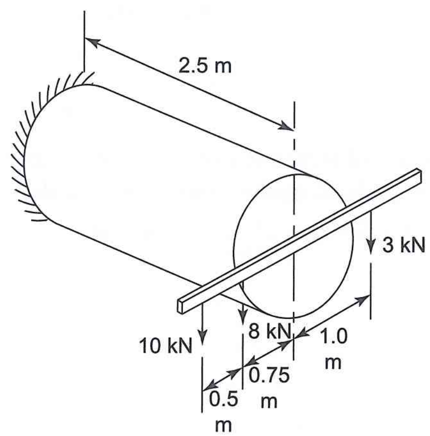
Ans. 8.6 kN, 23.9° down and to the left.



GURE P.2.3

.2.4. The circular section cylinder shown in Fig. P.2.4 is built-in at one end and carries a series of loads applied via a horizontal bar at its free end. Calculate the resultant downward force on the cylinder, the applied torque and the bending moment at its built-in end.

Ans. 21 kN, 15.5 kNm anticlockwise, 52.5 kNm.



GURE P.2.4

.2.5. Calculate the magnitude, inclination and point of action of the resultant of the system of non-concurrent forces shown in Fig. P.2.5. The coordinates of the points of action are given in metres.

Ans. 130.4 kN, 49.5° to the *x* direction at the point (0.81, 1.22).

.2.6. Calculate the support reactions in the beams shown in Fig. P.2.6(a)–(d).

Ans.

(a) $R_{A,H} = 9.2$ kN to left, $R_{A,V} = 6.9$ kN upwards, $R_B = 7.9$ kN upwards.

(b) $R_A = 65$ kN, $M_A = 400$ kNm anticlockwise.

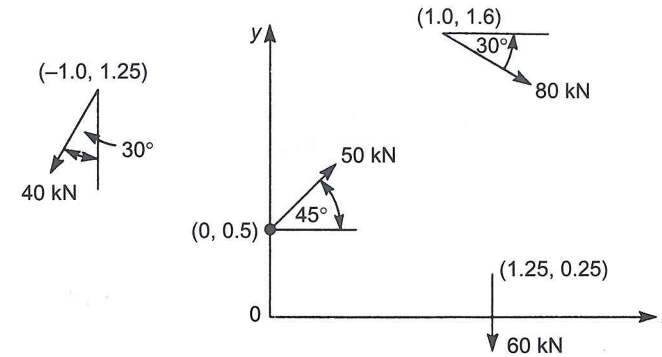


FIGURE P.2.5

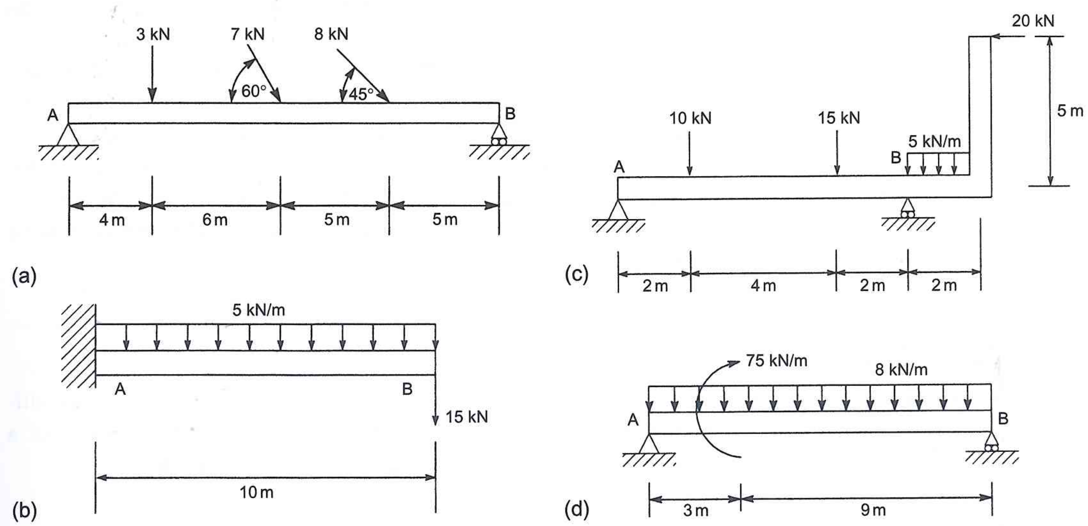


FIGURE P.2.6

P.2.7. Calculate the support reactions in the plane trusses shown in Fig. P.2.7(a) and (b).

Ans.

(a) $R_A = 57$ kN upwards, $R_B = 2$ kN downwards.

(b) $R_{A,H} = 3713.6$ N to left, $R_{A,V} = 835.6$ N downwards, $R_B = 4735.3$ N downwards.

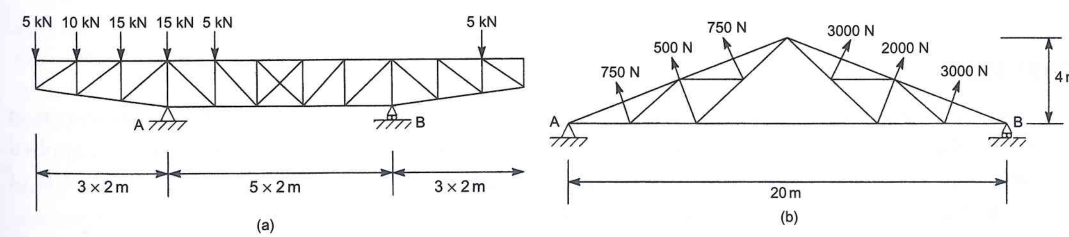


FIGURE P.2.7

CHAPTER

Normal Force, Shear Force, Bending Moment and Torsion

3

The purpose of a structure is to support the loads for which it has been designed. To accomplish this it must be able to transmit a load from one point to another, i.e. from the loading point to the supports. In Fig. 2.24, for example, the beam transmits the effects of the loads at B and C to the built-in end A. It achieves this by developing an *internal force* system and it is the distribution of these internal forces which must be determined before corresponding stress distributions and displacements can be found.

A knowledge of stress is essential in structural design where the cross-sectional area of a member must be such that stresses do not exceed values that would cause breakdown in the crystalline structure of the material of the member; in other words, a structural failure. In addition to stresses, strains, and thereby displacements, must be calculated to ensure that as well as strength a structural member possesses sufficient stiffness to prevent excessive distortions damaging surrounding portions of the complete structure.

In this chapter we shall examine the different types of load to which a structural member may be subjected and then determine corresponding internal force distributions.

3.1 Types of load

Structural members may be subjected to complex loading systems apparently comprised of several different types of load. However, no matter how complex such systems appear to be, they consist of a maximum of four basic load types: axial loads, shear loads, bending moments and torsion.

Axial load

Axial loads are applied along the longitudinal or centroidal axis of a structural member. If the action of the load is to increase the length of the member, the member is said to be in *tension* (Fig. 3.1(a)) and the applied load is *tensile*. A load that tends to shorten a member places the member in *compression* and is known as a *compressive* load (Fig. 3.1(b)). Members such as those shown in Fig. 3.1(a) and (b) are commonly found in pin-jointed frameworks where a member in tension is called a *tie* and one in compression a *strut* or *column*. More frequently, however, the name 'column' is associated with a vertical member carrying a compressive load, as illustrated in Fig. 3.1(c).

Shear load

Shear loads act perpendicularly to the axis of a structural member and have one of the forms shown in Fig. 3.2; in this case the members are *beams*. Figure 3.2(a) shows a *concentrated* shear load, W, applied to a cantilever beam. The shear load in Fig. 3.2(b) is *distributed* over a length of the beam and is of *intensity* w (force units) per unit length (see Section 1.7).

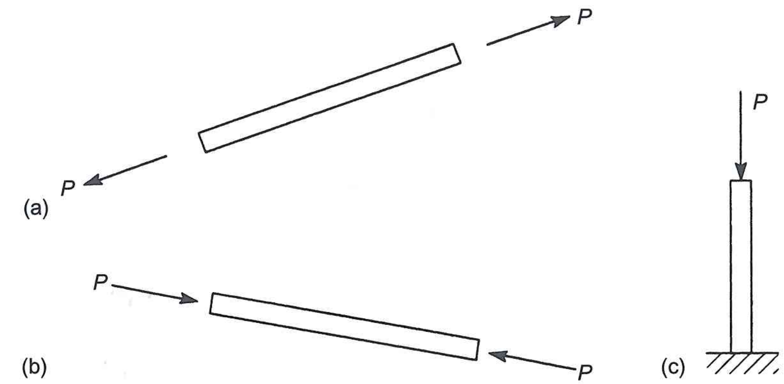


FIGURE 3.1

Axially loaded members.

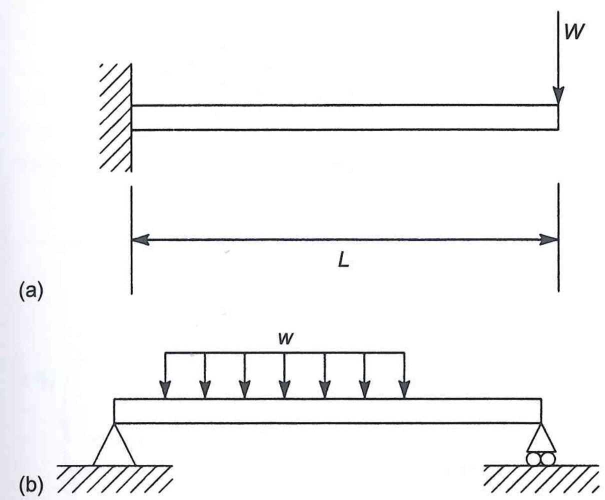


FIGURE 3.2

Shear loads applied to beams.

Bending moment

In practice it is difficult to apply a pure bending moment such as that shown in Fig. 3.3(a) to a beam. Generally, pure bending moments arise through the application of other types of load to adjacent structural members. For example, in Fig. 3.3(b), a vertical member BC is attached to the cantilever AB and carries a horizontal shear load, P (as far as BC is concerned). AB is therefore subjected to a pure moment, $M = Ph$, at B together with an axial load, P.

Torsion

A similar situation arises in the application of a pure torque, T (Fig. 3.4(a)), to a beam. A practical example of a torque applied to a cantilever beam is given in Fig. 3.4(b) where the horizontal member BC supports a vertical shear load at C. The cantilever AB is then subjected to a pure torque, $T = Wh$, plus a shear load, W.

All the loads illustrated in Figs 3.1–3.4 are applied to the various members by some external agency and are therefore *externally applied loads*. Each of these loads induces reactions in the support systems of the different beams; examples of the calculation of support reactions are given in Section 2.5. Since structures are in equilibrium under a force system of externally applied loads and support reactions, it follows that the support reactions are themselves externally applied loads.

Now consider the cantilever beam of Fig. 3.2(a). If we were to physically cut through the beam at some section 'mm' (Fig. 3.5(a)) the portion BC would no longer be able to support the load, W. The portion AB of the beam therefore performs the same function for the portion BC as does the wall for the complete beam. Thus at the section mm the portion AB applies a force W and a moment M to the

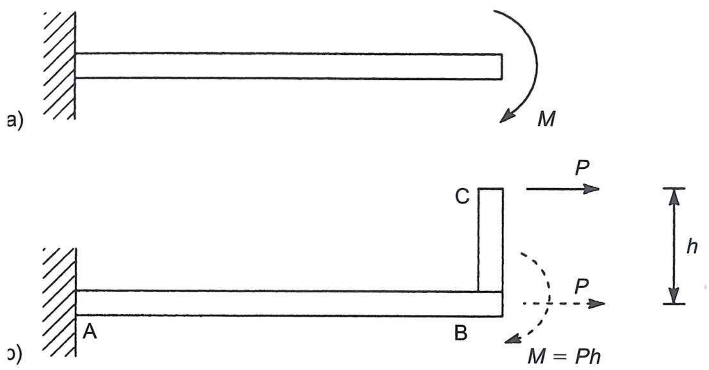
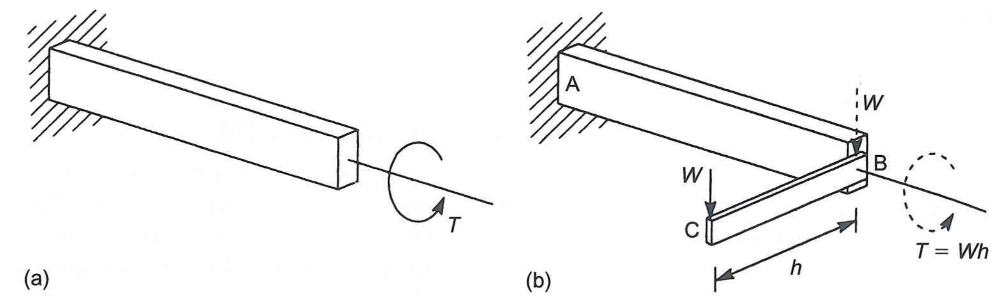


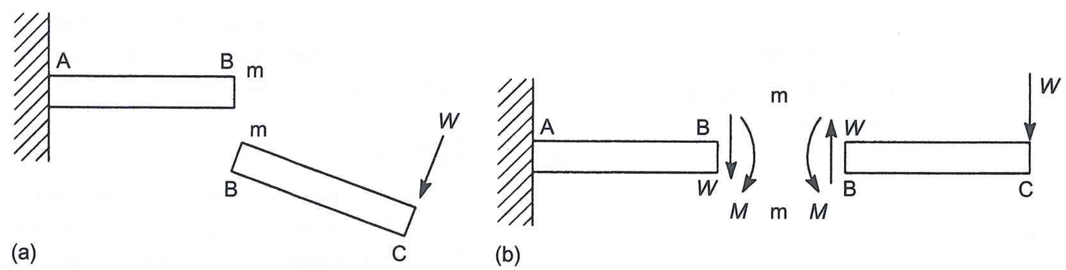
FIGURE 3.3

Moments applied to beams.



ꟷIGURE 3.4

ɔrques applied to a beam.



ꟷGURE 3.5

ıternal force system generated by an external shear load.

Newton's Third Law of Motion), BC exerts an equal force system on AB, but opposite in direction. he complete force systems acting on the two faces of the section mm are shown in Fig. 3.5(b).

Systems of forces such as those at the section mm are known as *internal forces*. Generally, they vary ıroughout the length of a structural member as can be seen from Fig. 3.5(b) where the internal ıoment, *M*, increases in magnitude as the built-in end is approached due to the increasing rotational ffect of *W*. We note that applied loads of one type can induce internal forces of another. For example, ı Fig. 3.5(b) the external shear load, *W*, produces both shear and bending at the section mm.

Internal forces are distributed throughout beam sections in the form of stresses. It follows that the

are therefore often known as *stress resultants*. However, before an individual stress distribution can be found it is necessary to determine the corresponding internal force. Also, in design problems, it is necessary to determine the position and value of maximum stress and displacement. Usually, the first step in the analysis of a structure is to calculate the distribution of each of the four basic internal force types throughout the component structural members. We shall therefore determine the distributions of the four internal force systems in a variety of structural members. First, however, we shall establish a notation and sign convention for each type of force.

3.2 Notation and sign convention

We shall be concerned initially with structural members having at least one longitudinal plane of symmetry. Normally this will be a vertical plane and will contain the externally applied loads. Later, however, we shall investigate the bending and shear of beams having unsymmetrical sections so that as far as possible the notation and sign convention we adopt now will be consistent with that required later.

The axes system we shall use is the right-handed system shown in Fig. 3.6 in which the x axis is along the longitudinal axis of the member and the y axis is vertically upwards. Externally applied loads W (concentrated) and w (distributed) are shown acting vertically downwards since this is usually the situation in practice. In fact, choosing a sign convention for these externally applied loads is not particularly important and can be rather confusing since they will generate support reactions, which are external loads themselves, in an opposite sense. An external axial load P is positive when tensile and a torque T is positive if applied in an anticlockwise sense when viewed in the direction xO. Later we shall be concerned with displacements in structural members and here the vertical displacement v is positive in the positive direction of the y axis.

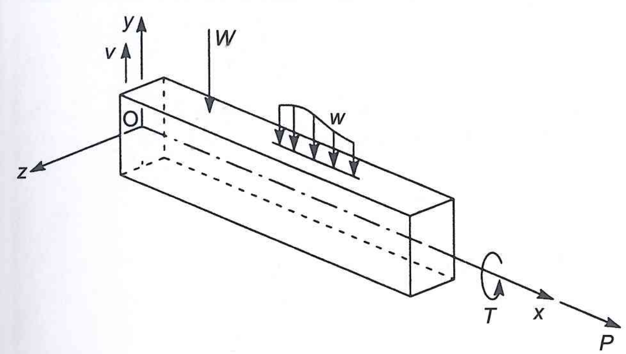


FIGURE 3.6

Notation and sign conventions for displacements and externally applied loads.

We have seen that external loads generate internal force systems and for these it is essential to adopt a sign convention since, unless their directions and senses are known, it is impossible to calculate stress distributions.

Figure 3.7 shows a positive set of internal forces acting at two sections of a beam.

Note that the forces and moments acting on opposite faces of a section are identical and act in opposite directions

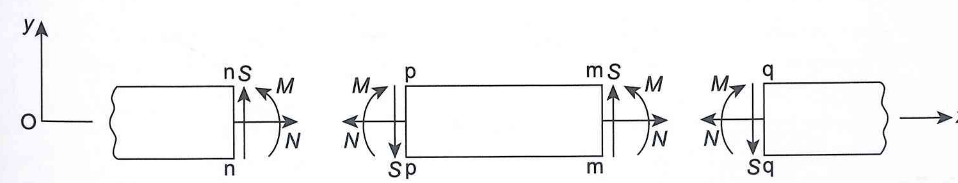


FIGURE 3.7

ince the internal equilibrium of the beam must be maintained. If this were not the case one part of the eam would part company with the other. A difficulty now arises in that a positive internal force, say he shear force S, acts upwards on one face of a section and downwards on the opposite face. We must herefore specify the face of the section we are considering. We can do this by giving signs to the differnt faces. In Fig. 3.7 we define a *positive face* as having an outward normal in the positive direction of he x axis (faces nn and mm) and a *negative face* as having an outward normal in the negative direction f the x axis (faces pp and qq). At nn and mm positive internal forces act in positive directions on positive faces while at pp and qq positive internal forces act in negative directions on negative faces.

A positive bending moment M, clockwise on the negative face pp and anticlockwise on the positive ace mm, will cause the upper surface of the beam to become concave and the lower surface convex. This, for obvious reasons, is called a *sagging* bending moment. A negative bending moment will proluce a convex upper surface and a concave lower one and is therefore termed a *hogging* bending noment.

The axial, or normal, force N is positive when tensile, i.e. it pulls away from either face of a section, nd a positive internal torque T is anticlockwise on positive internal faces.

Generally the structural engineer will need to know peak values of these internal forces in a strucural member. To determine these peak values *internal force diagrams* are constructed; the methods will e illustrated by examples.

.3 Normal force

EXAMPLE 3.1

Construct a normal force diagram for the beam AB shown in Fig. 3.8(a).

The first step is to calculate the support reactions using the methods described in Section 2.5. In this case, since the beam is on a roller support at B, the horizontal load at B is reacted at A; clearly $R_{A,H} = 10$ kN acting to the left.

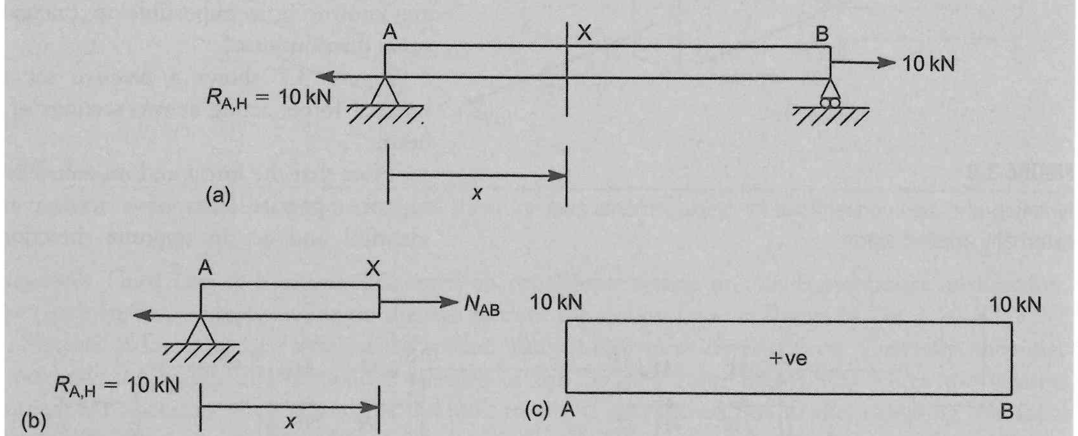


FIGURE 3.8

Normal force diagram for the beam of Ex. 3.1.

Generally the distribution of an internal force will change at a loading discontinuity. In this case there is no loading discontinuity at any section of the beam so that we can determine the complete distribution of the normal force by calculating the normal force at any section X, a distance x from A.

Consider the length AX of the beam as shown in Fig. 3.8(b) (equally we could consider the length XB). The internal normal force acting at X is N_{AB} which is shown acting in a positive (tensile) direction. The length AX of the beam is in equilibrium under the action of $R_{A,H}$ (=10 kN) and N_{AB}. Thus, from Section 2.4, for equilibrium in the x direction

$$N_{AB} - R_{A,H} = N_{AB} - 10 = 0$$

which gives

$$N_{AB} = +\,10 \text{ kN}$$

N_{AB} is positive and therefore acts in the assumed positive direction; the normal force diagram for the complete beam is then as shown in Fig. 3.8(c).

When the equilibrium of a portion of a structure is considered as in Fig. 3.8(b) we are using what is termed a *free body diagram.*

EXAMPLE 3.2

Draw a normal force diagram for the beam ABC shown in Fig. 3.9(a).

Again by considering the overall equilibrium of the beam we see that $R_{A,H} = 10$ kN acting to the left (C is the roller support).

In this example there is a loading discontinuity at B so that the distribution of the normal force in AB will be different to that in BC. We must therefore determine the normal force at an arbitrary section X_1 between A and B, and then at an arbitrary section X_2 between B and C.

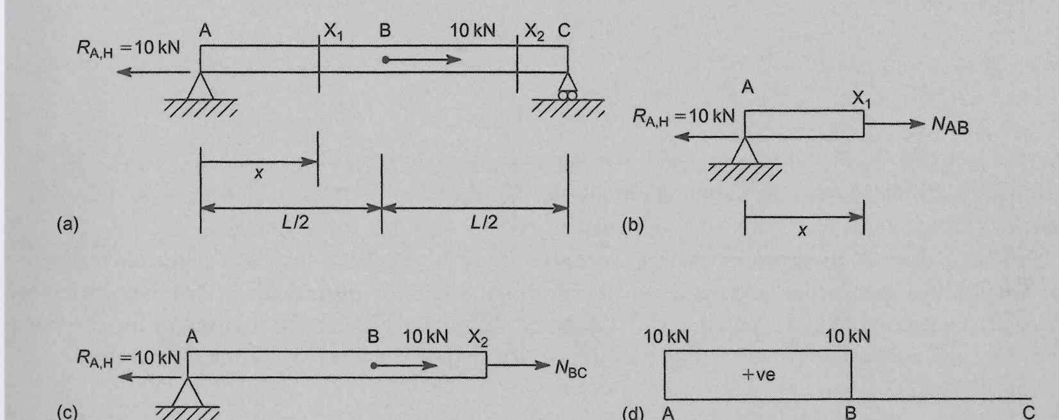


FIGURE 3.9

Normal force diagram for the beam of Ex. 3.2.

The free body diagram for the portion of the beam AX_1 is shown in Fig. 3.9(b). (Alternatively we could consider the portion X_1C). As before, we draw in a positive normal force, N_{AB}. Then, for equilibrium of AX_1 in the x direction

$$N_{AB} - 10 = 0$$

so that

$$N_{AB} = +10 \text{ kN (tension)}$$

Now consider the length ABX_2 of the beam; again we draw in a positive normal force, N_{BC}. Then for equilibrium of ABX_2 in the x direction

$$N_{BC} + 10 - 10 = 0$$

which gives

$$N_{BC} = 0$$

Note that we would have obtained the same result by considering the portion X_2C of the beam. Finally the complete normal force diagram for the beam is drawn as shown in Fig. 3.9(d).

EXAMPLE 3.3

Figure 3.10(a) shows a beam ABCD supporting three concentrated loads, two of which are inclined to the longitudinal axis of the beam. Construct the normal force diagram for the beam and determine the maximum value.

In this example we are only concerned with determining the normal force distribution in the beam, so that it is unnecessary to calculate the vertical reactions at the supports. Further, the horizontal components of the inclined loads can only be resisted at A since D is a roller support. Thus, considering the horizontal equilibrium of the beam

$$R_{A,H} + 6\cos 60^\circ - 4\cos 60^\circ = 0$$

which gives

$$R_{A,H} = -1 \text{ kN}$$

The negative sign of $R_{A,H}$ indicates that the reaction acts to the right and not to the left as originally assumed. However, rather than change the direction of $R_{A,H}$ in the diagram, it is simpler to retain the assumed direction and then insert the negative value as required.

Although there is an apparent loading discontinuity at B, the 2 kN load acts perpendicularly to the longitudinal axis of the beam and will therefore not affect the normal force. We may therefore consider the normal force at any section X_1 between A and C. The free body diagram for the portion AX_1 of the beam is shown in Fig. 3.10(b); again we draw in a positive normal force N_{AC}. For equilibrium of AX_1

$$N_{AC} - R_{A,H} = 0$$

so that

$$N_{AC} = R_{A,H} = -1 \text{ kN (compression)}$$

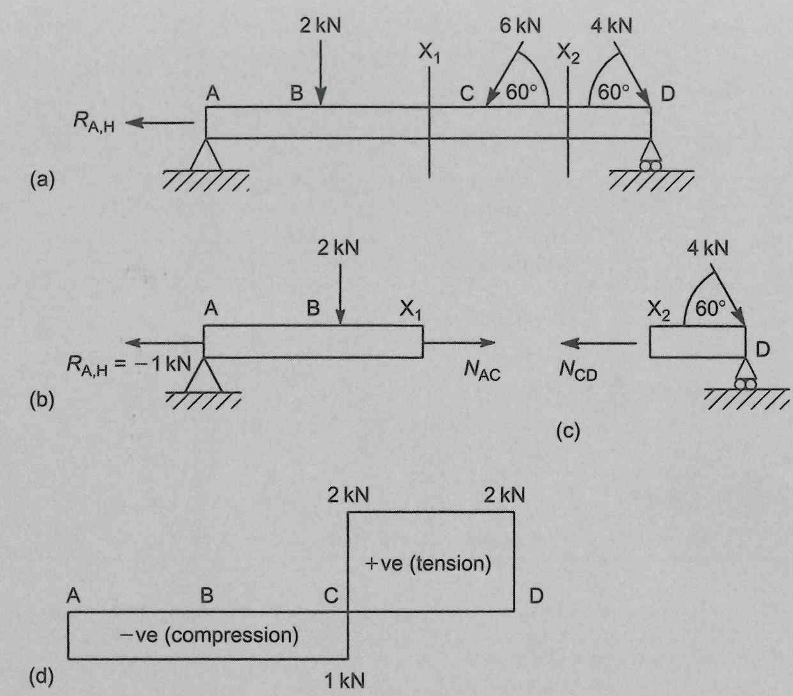


FIGURE 3.10

Normal force diagram for the beam of Ex. 3.3.

The horizontal component of the inclined load at C produces a loading discontinuity so that we now consider the normal force at any section X_2 between C and D. Here it is slightly simpler to consider the equilibrium of the length X_2D of the beam rather than the length AX_2. Thus, from Fig. 3.10(c)

$$N_{CD} - 4\cos 60^\circ = 0$$

which gives

$$N_{CD} = +2 \text{ kN (tension)}$$

From the completed normal force diagram in Fig. 3.10(d) we see that the maximum normal force in the beam is 2 kN (tension) acting at all sections between C and D.

EXAMPLE 3.4

Construct the normal force diagram for the cranked cantilever beam shown in Fig. 3.11(a).

Note that in this example there will be two components of support reaction at the built-in end of the beam, $R_{A,H}$, and $R_{A,V}$ (there will also be a moment reaction but since we are concerned only with normal force this is irrelevant). However, if we consider the equilibrium of portions of the beam away from the built-in end it will not be necessary to calculate them. Note also that there is a

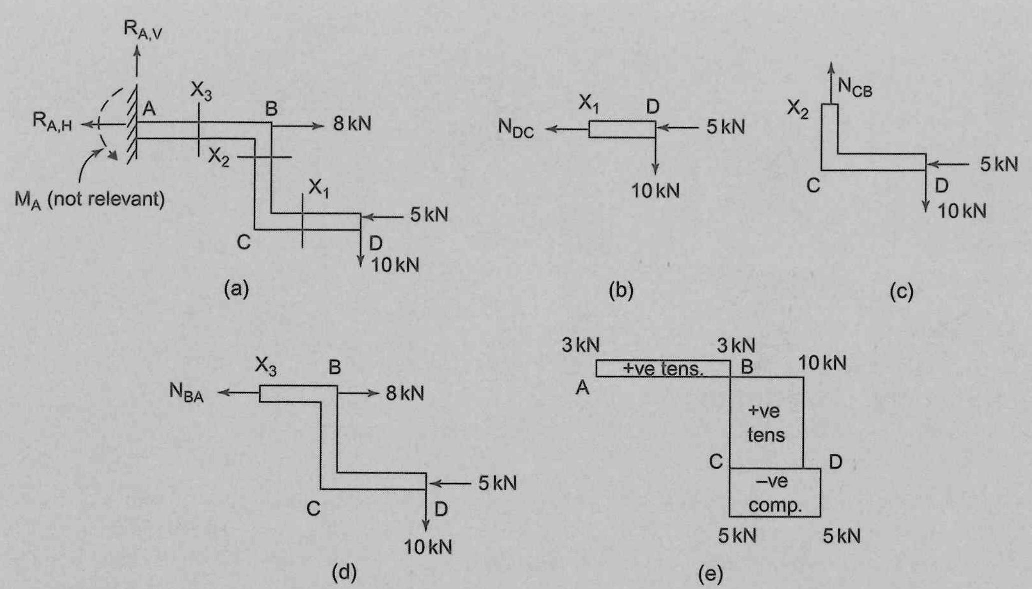


FIGURE 3.11

Normal force diagram for the beam of Ex. 3.4.

loading discontinuity at B and structural discontinuities at C and B. Initially, therefore, we consider the normal force, N_{DC}, at the section X_1 as shown in Fig. 3.11(b).

For horizontal equilibrium of the length DX_1 of the beam

$$N_{DC} + 5 = 0$$

so that

$$N_{DC} = -5 \text{ kN (compression)}$$

The vertical 10 kN load acting at D will produce a normal force in CB. Then, considering the vertical equilibrium of the portion DCX_2 of the beam in Fig. 3.11(c)

$$N_{CB} - 10 = 0$$

which gives

$$N_{CB} = +10 \text{ kN (tension)}$$

Finally we consider the horizontal equilibrium of the portion $DCBX_3$ of the beam in Fig. 3.11(d).

$$N_{BA} - 8 + 5 = 0$$

from which

$$N_{BA} = +3 \text{ kN (tension)}$$

The normal force diagram for the complete beam is then as shown in Fig. 3.11(e). The normal force for the vertical portion CB may be drawn on either side of CB as is convenient.

3.4 Shear force and bending moment

It is convenient to consider shear force and bending moment distributions in beams simultaneously since, as we shall see in Section 3.5, they are directly related. Again the method of construction of shear force and bending moment diagrams will be illustrated by examples.

EXAMPLE 3.5

Cantilever beam with a concentrated load at the free end (Fig. 3.12).

Generally, as in the case of normal force distributions, we require the variation in shear force and bending moment along the length of a beam. Again, loading discontinuities, such as concentrated loads and/or a sudden change in the intensity of a distributed load, cause discontinuities in the distribution of shear force and bending moment so that it is necessary to consider a series of sections, one between each loading discontinuity. In this example, however, there are no loading discontinuities between the built-in end A and the free end B so that we may consider a section X at any point between A and B.

For many beams the value of each support reaction must be calculated before the shear force and bending moment distributions can be obtained. In Fig. 3.12(a) a consideration of the overall equilibrium of the beam (see Section 2.5) gives a vertical reaction, *W*, and a moment reaction, *WL*, at

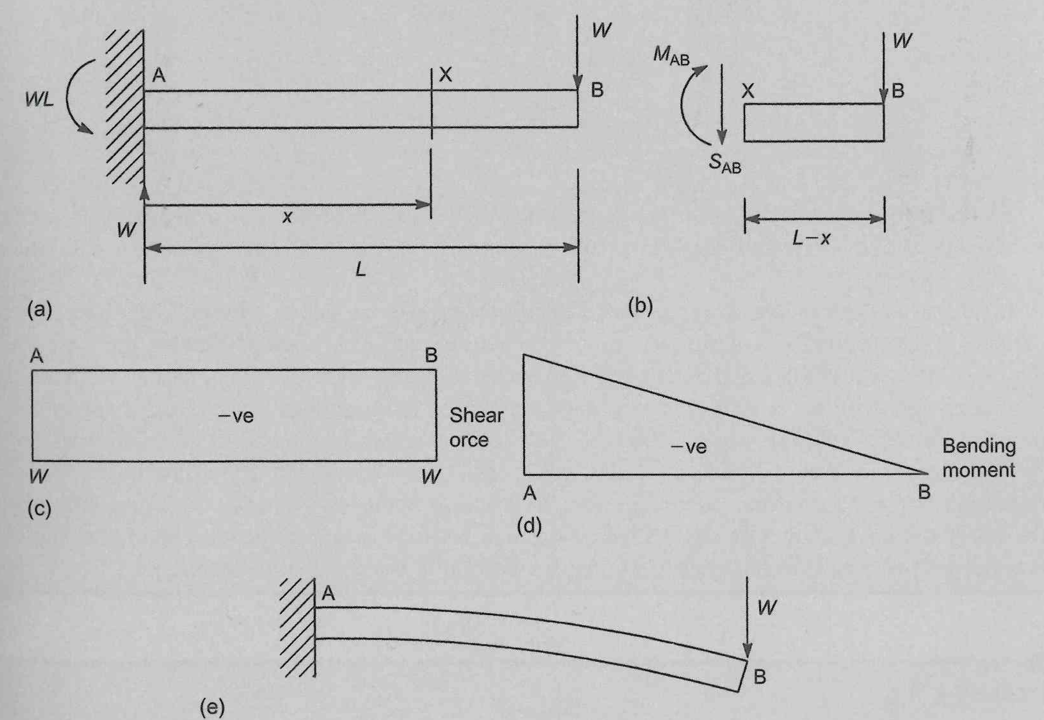


FIGURE 3.12

Shear force and bending moment diagrams for the beam of Ex. 3.5.

the built-in end. However, if we consider the equilibrium of the length XB of the beam as shown in the free body diagram in Fig. 3.12(b), this calculation is unnecessary.

As in the case of normal force distributions we assign positive directions to the shear force, S_{AB}, and bending moment, M_{AB}, at the section X. Then, for vertical equilibrium of the length XB of the beam we have

$$S_{AB} + W = 0$$

which gives

$$S_{AB} = -W$$

The shear force is therefore constant along the length of the beam and the shear force diagram is rectangular in shape, as shown in Fig. 3.12(c).

The bending moment, M_{AB}, is now found by considering the moment equilibrium of the length XB of the beam about the section X. Alternatively we could take moments about B, but this would involve the moment of the shear force, S_{AB}, about B. This approach, although valid, is not good practice since it includes a previously calculated quantity; in some cases, however, this is unavoidable. Thus, taking moments about the section X we have

$$M_{AB} + W(L - x) = 0$$

so that

$$M_{AB} = -W(L - x) \qquad \text{(i)}$$

Equation (i) shows that M_{AB} varies linearly along the length of the beam, is negative, i.e. hogging, at all sections and increases from zero at the free end ($x = L$) to $-WL$ at the built-in end where $x = 0$.

It is usual to draw the bending moment diagram on the tension side of a beam. This procedure is particularly useful in the design of reinforced concrete beams since it shows directly the surface of the beam near which the major steel reinforcement should be provided. Also, drawing the bending moment diagram on the tension side of a beam can give an indication of the deflected shape as illustrated in Exs 3.5–3.8. This is not always the case, however, as we shall see in Exs 3.9 and 3.10.

In this case the beam will bend as shown in Fig. 3.12(e), so that the upper surface of the beam is in tension and the lower one in compression; the bending moment diagram is therefore drawn on the upper surface as shown in Fig. 3.12(d). Note that negative (hogging) bending moments applied in a vertical plane will always result in the upper surface of a beam being in tension.

EXAMPLE 3.6

Cantilever beam carrying a uniformly distributed load of intensity w.

Again it is unnecessary to calculate the reactions at the built-in end of the cantilever; their values are, however, shown in Fig. 3.13(a). Note that for the purpose of calculating the moment reaction

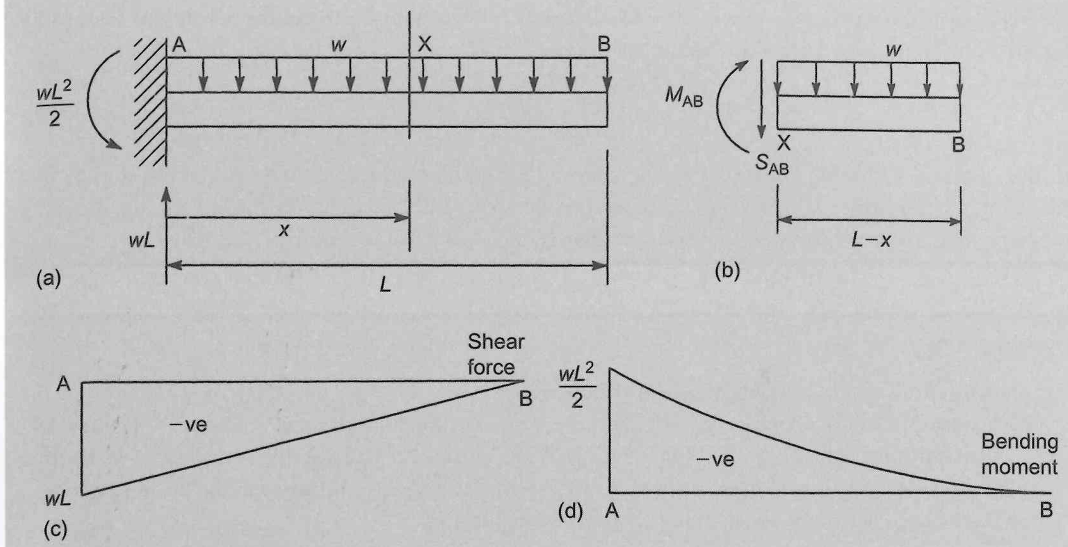


FIGURE 3.13

Shear force and bending moment diagrams for the beam of Ex. 3.6.

the uniformly distributed load may be replaced by a concentrated load ($= wL$) acting at a distance $L/2$ from A.

There is no loading discontinuity between A and B so that we may consider the shear force and bending moment at any section X between A and B. As before, we insert positive directions for the shear force, S_{AB}, and bending moment, M_{AB}, in the free body diagram of Fig. 3.13(b). Then, for vertical equilibrium of the length XB of the beam

$$S_{AB} + w(L - x) = 0$$

so that

$$S_{AB} = -w(L - x) \qquad \text{(i)}$$

Therefore S_{AB} varies linearly with x and varies from zero at B to $-wL$ at A (Fig. 3.13(c)).

Now consider the moment equilibrium of the length AB of the beam and take moments about X

$$M_{AB} + \frac{w}{2}(L - x)^2 = 0$$

which gives

$$M_{AB} = -\frac{w}{2}(L - x)^2 \qquad \text{(ii)}$$

Note that the total load on the length XB of the beam is $w(L - x)$, which we may consider acting as a concentrated load at a distance $(L - x)/2$ from X. From Eq. (ii) we see that the bending moment, M_{AB}, is negative at all sections of the beam and varies parabolically as shown in Fig. 3.13(d) where the bending moment diagram is again drawn on the tension side of the beam. The actual shape of the bending moment diagram may be found by plotting values or, more conveniently, by examining Eq. (ii). Differentiating with respect to x we obtain

$$\frac{dM_{AB}}{dx} = w(L - x) \qquad \text{(iii)}$$

so that when $x = L$, $dM_{AB}/dx = 0$ and the bending moment diagram is tangential to the datum line AB at B. Furthermore it can be seen from Eq. (iii) that the gradient (dM_{AB}/dx) of the bending moment diagram decreases as x increases, so that its shape is as shown in Fig. 3.13(d).

EXAMPLE 3.7

Simply supported beam carrying a central concentrated load.

In this example it is necessary to calculate the value of the support reactions, both of which are seen, from symmetry, to be $W/2$ (Fig. 3.14(a)). Also, there is a loading discontinuity at B, so that we must consider the shear force and bending moment first at an arbitrary section X_1 say, between A and B and then at an arbitrary section X_2 between B and C.

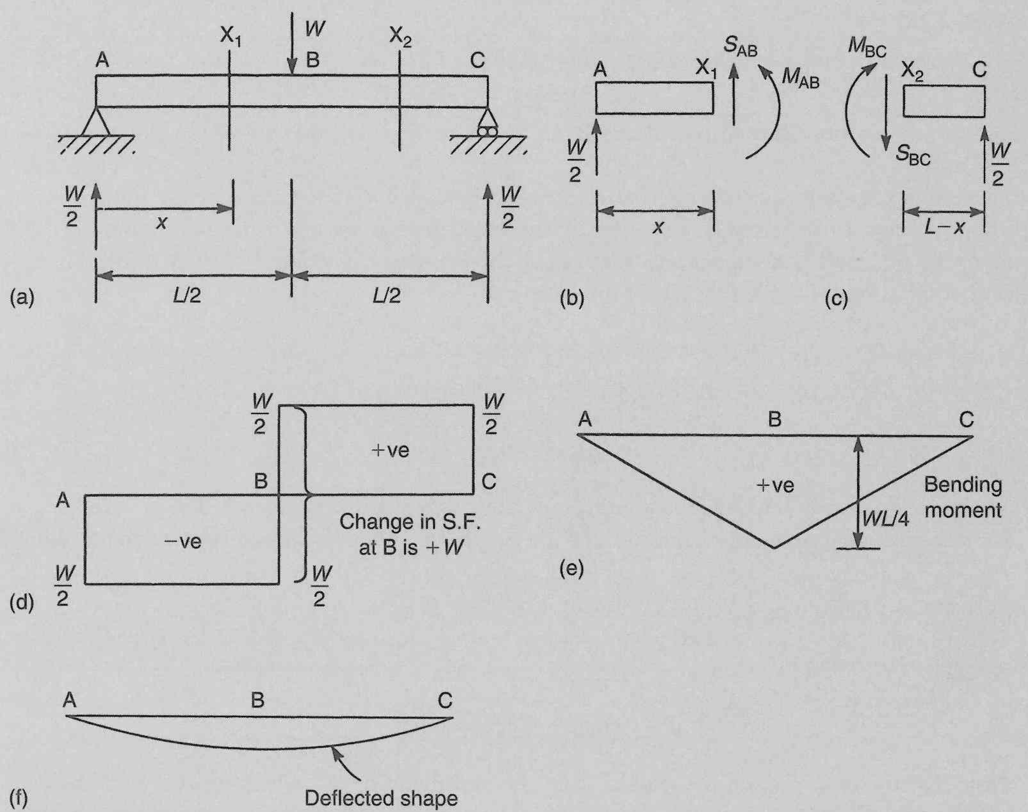


FIGURE 3.14

Shear force and bending moment diagrams for the beam of Ex. 3.7.

From the free body diagram in Fig. 3.14(b) in which both S_{AB} and M_{AB} are in positive directions we see, by considering the vertical equilibrium of the length AX_1 of the beam, that

$$S_{AB} + \frac{W}{2} = 0$$

which gives

$$S_{AB} = -\frac{W}{2}$$

S_{AB} is therefore constant at all sections of the beam between A and B, in other words, from a section immediately to the right of A to a section immediately to the left of B.

Now consider the free body diagram of the length X_2C of the beam in Fig. 3.14(c). Note that, equally, we could have considered the length ABX_2, but this would have been slightly more complicated in terms of the number of loads acting. For vertical equilibrium of X_2C

$$S_{BC} - \frac{W}{2} = 0$$

from which

$$S_{BC} = +\frac{W}{2}$$

and we see that S_{BC} is constant at all sections of the beam between B and C so that the complete shear force diagram has the form shown in Fig. 3.14(d). Note that the *change* in shear force from that at a section immediately to the left of B to that at a section immediately to the right of B is $+W$. We shall consider the implications of this later in the chapter.

It would also appear from Fig. 3.14(d) that there are two different values of shear force at the same section B of the beam. This results from the assumption that W is concentrated at a point which, practically, is impossible since there would then be an infinite bearing pressure on the surface of the beam. In practice, the load W and the support reactions would be distributed over a small length of beam (Fig. 3.15(a)) so that the actual shear force distribution would be that shown in Fig. 3.15(b).

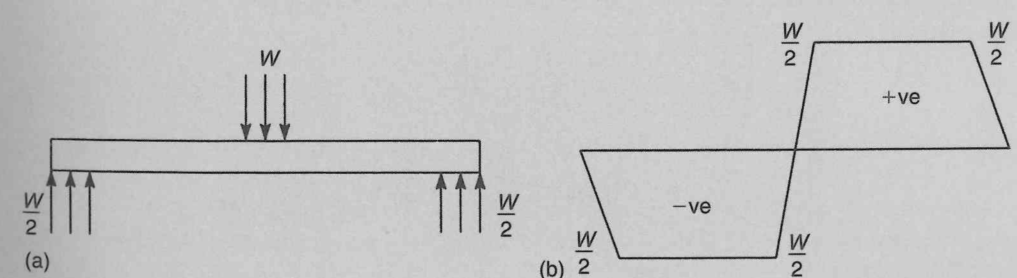


FIGURE 3.15

Shear force diagram in a practical situation.

The distribution of the bending moment in AB is now found by considering the moment equilibrium about X_1 of the length AX_1 of the beam in Fig. 3.14(b). Thus

$$M_{AB} - \frac{W}{2}x = 0$$

or

$$M_{AB} = \frac{W}{2}x \tag{i}$$

Therefore M_{AB} varies linearly from zero at A ($x = 0$) to $+WL/4$ at B ($x = L/2$).

Now considering the length X_2C of the beam in Fig. 3.14(c) and taking moments about X_2.

$$M_{BC} - \frac{W}{2}(L - x) = 0$$

which gives

$$M_{BC} = +\frac{W}{2}(L - x) \tag{ii}$$

From Eq. (ii) we see that M_{BC} varies linearly from $+WL/4$ at B ($x = L/2$) to zero at C($x = L$).

The complete bending moment diagram is shown in Fig. 3.14(e). Note that the bending moment is positive (sagging) at all sections of the beam so that the lower surface of the beam is in tension. In this example the deflected shape of the beam would be that shown in Fig. 3.14(f).

EXAMPLE 3.8

Simply supported beam carrying a uniformly distributed load.

The symmetry of the beam and its load may again be used to determine the support reactions which are each $wL/2$. Furthermore, there is no loading discontinuity between the ends A and B of the beam so that it is sufficient to consider the shear force and bending moment at just one section X, a distance x, say, from A; again we draw in positive directions for the shear force and bending moment at the section X in the free body diagram shown in Fig. 3.16(b).

Considering the vertical equilibrium of the length AX of the beam gives

$$S_{AB} - wx + w\frac{L}{2} = 0$$

i.e.

$$S_{AB} = +w\left(x - \frac{L}{2}\right) \tag{i}$$

S_{AB} therefore varies linearly along the length of the beam from $-wL/2$ at A ($x = 0$) to $+wL/2$ at B ($x = L$). Note that $S_{AB} = 0$ at mid-span ($x = L/2$).

Now taking moments about X for the length AX of the beam in Fig. 3.16(b) we have

$$M_{AB} + \frac{wx^2}{2} - \frac{wL}{2}x = 0$$

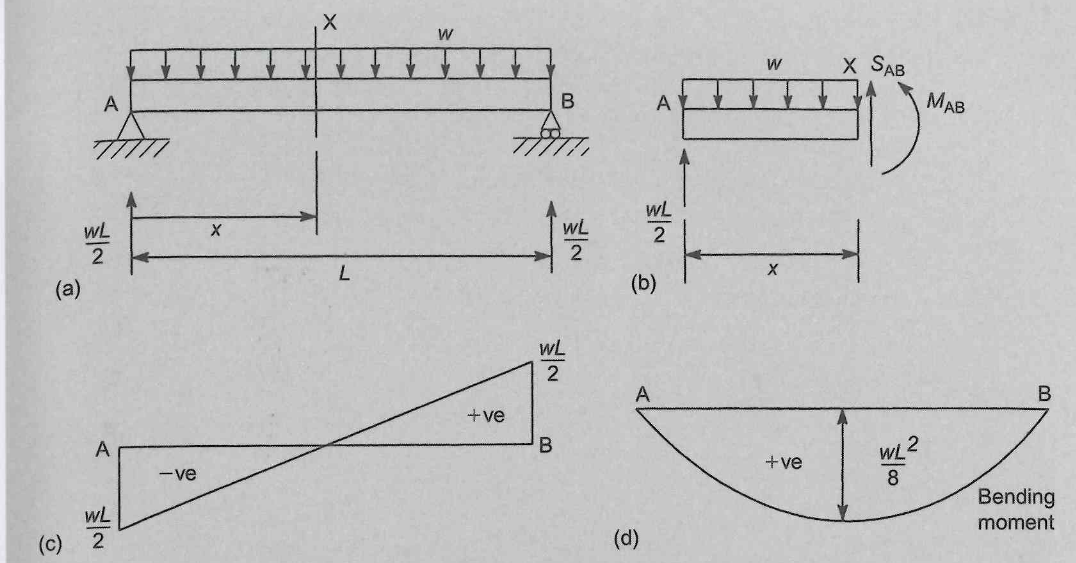


FIGURE 3.16

Shear force and bending moment diagrams for the beam of Ex. 3.8.

from which

$$M_{AB} = +\frac{wx}{2}(L - x) \tag{ii}$$

Thus M_{AB} varies parabolically along the length of the beam and is positive (sagging) at all sections of the beam except at the supports ($x = 0$ and $x = L$) where it is zero.

Also, differentiating Eq. (ii) with respect to x gives

$$\frac{dM_{AB}}{dx} = w\left(\frac{L}{2} - x\right) \tag{iii}$$

From Eq. (iii) we see that $dM_{AB}/dx = 0$ at mid-span where $x = L/2$, so that the bending moment diagram has a turning value or mathematical maximum at this section. In this case this mathematical maximum is the maximum value of the bending moment in the beam and is, from Eq. (ii), $+wL^2/8$.

The bending moment diagram for the beam is shown in Fig. 3.16(d) where it is again drawn on the tension side of the beam; the deflected shape of the beam will be identical in form to the bending moment diagram.

Examples 3.5–3.8 may be regarded as 'standard' cases and it is useful to memorize the form that the shear force and bending moment diagrams take including the principal values.

EXAMPLE 3.9

Simply supported beam with cantilever overhang (Fig. 3.17(a)).

The support reactions are calculated using the methods described in Section 2.5. Thus, taking moments about B in Fig. 3.17(a) we have

$$R_A \times 2 - 2 \times 3 \times 0.5 + 1 \times 1 = 0$$

which gives

$$R_A = 1 \text{ kN}$$

From vertical equilibrium

$$R_B + R_A - 2 \times 3 - 1 = 0$$

so that

$$R_B = 6 \text{ kN}$$

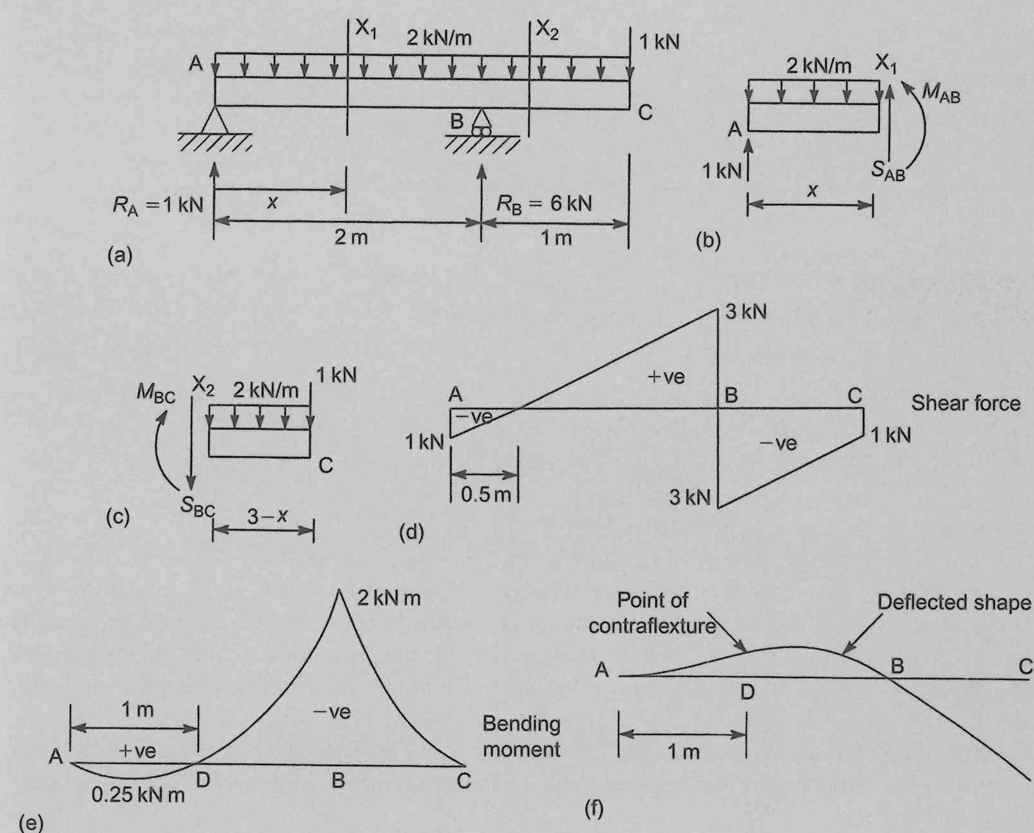


FIGURE 3.17

Shear force and bending moment diagrams for the beam of Ex. 3.9.

The support reaction at B produces a loading discontinuity at B so that we must consider the shear force and bending moment at two arbitrary sections of the beam, X_1 in AB and X_2 in BC. Free body diagrams are therefore drawn for the lengths AX_1 and X_2C of the beam and positive directions for the shear force and bending moment drawn in as shown in Fig. 3.17(b) and (c). Alternatively, we could have considered the lengths X_1BC and ABX_2, but this approach would have involved slightly more complicated solutions in terms of the number of loads applied.

Now from the vertical equilibrium of the length AX_1 of the beam in Fig. 3.17(b) we have

$$S_{AB} - 2x + 1 = 0$$

or

$$S_{AB} = 2x - 1 \tag{i}$$

The shear force therefore varies linearly in AB from -1 kN at A($x = 0$) to $+3$ kN at B($x = 2$ m). Note that $S_{AB} = 0$ at $x = 0.5$ m.

Consideration of the vertical equilibrium of the length X_2C of the beam in Fig. 3.17(c) gives

$$S_{BC} + 2(3 - x) + 1 = 0$$

from which

$$S_{BC} = 2x - 7 \tag{ii}$$

Equation (ii) shows that S_{BC} varies linearly in BC from -3 kN at B($x = 2$ m) to -1 kN at C($x = 3$ m).

The complete shear force diagram for the beam is shown in Fig. 3.17(d).

The bending moment, M_{AB}, is now obtained by considering the moment equilibrium of the length AX_1 of the beam about X_1 in Fig. 3.17(b). Hence

$$M_{AB} + 2x\frac{x}{2} - 1x = 0$$

so that

$$M_{AB} = x - x^2 \tag{iii}$$

which is a parabolic function of x. The distribution may be plotted by selecting a series of values of x and calculating the corresponding values of M_{AB}. However, this would not necessarily produce accurate estimates of either the magnitudes and positions of the maximum values of M_{AB} or, say, the positions of the zero values of M_{AB} which, as we shall see later, are important in beam design. A better approach is to examine Eq. (iii) as follows. Clearly when $x = 0$, $M_{AB} = 0$ as would be expected at the simple support at A. Also at B, where $x = 2$ m, $M_{AB} = -2$ kN so that although the support at B is a simple support and allows rotation of the beam, there is a moment at B; this is produced by the loads on the cantilever overhang BC. Rewriting Eq. (iii) in the form

$$M_{AB} = x(1 - x) \tag{iv}$$

we see immediately that $M_{AB} = 0$ at $x = 0$ (as demonstrated above) and that $M_{AB} = 0$ at $x = 1$ m, the point D in Fig. 3.17(e). We shall see later in Chapter 9 that at the point in the beam where the bending moment changes sign the curvature of the beam is zero; this point is known as a *point of contraflexure* or *point of inflection*. Now differentiating Eq. (iii) with respect to x we obtain

$$\frac{dM_{AB}}{dx} = 1 - 2x \tag{v}$$

and we see that $dM_{AB}/dx = 0$ at $x = 0.5$ m. In other words M_{AB} has a turning value or mathematical maximum at $x = 0.5$ m at which point $M_{AB} = 0.25$ kN m. Note that this is not the greatest value of bending moment in the span AB. Also it can be seen that for $0 < x < 0.5$ m, dM_{AB}/dx decreases with x while for $0.5\text{ m} < x < 2$ m, dM_{AB}/dx increases negatively with x.

Now we consider the moment equilibrium of the length X_2C of the beam in Fig. 3.17(c) about X_2

$$M_{BC} + \frac{2}{2}(3-x)^2 + 1(3-x) = 0$$

so that

$$M_{BC} = -12 + 7x - x^2 \tag{vi}$$

from which we see that dM_{BC}/dx is not zero at any point in BC and that as x increases dM_{BC}/dx decreases.

The complete bending moment diagram is therefore as shown in Fig. 3.17(e). Note that the value of zero shear force in AB coincides with the turning value of the bending moment.

In this particular example it is not possible to deduce the displaced shape of the beam from the bending moment diagram. Only three facts relating to the displaced shape can be stated with certainty; these are, the deflections at A and B are zero and there is a point of contraflexure at D, 1 m from A. However, using the method described in Section 13.2 gives the displaced shape shown in Fig. 3.17(f). Note that, although the beam is subjected to a sagging bending moment over the length AD, the actual deflection is upwards; clearly this could not have been deduced from the bending moment diagram.

EXAMPLE 3.10

Simply supported beam carrying a point moment.

From a consideration of the overall equilibrium of the beam (Fig. 3.18(a)) the support reactions are $R_A = M_0/L$ acting vertically upward and $R_C = M_0/L$ acting vertically downward. Note that R_A and R_C are independent of the point of application of M_0.

Although there is a loading discontinuity at B it is a point moment and will not affect the distribution of shear force. Thus, by considering the vertical equilibrium of either AX_1 in Fig. 3.18(b) or X_2C in Fig. 3.18(c) we see that

$$S_{AB} = S_{BC} = -\frac{M_0}{L} \tag{i}$$

The shear force is therefore constant along the length of the beam as shown in Fig. 3.18(d).

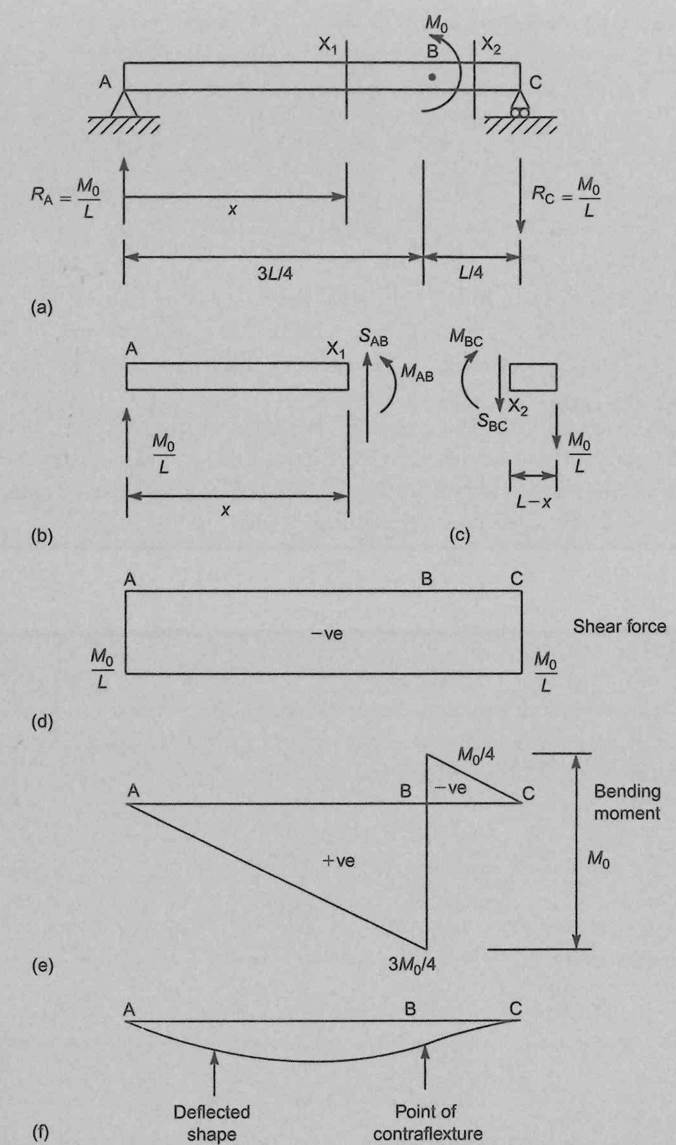


FIGURE 3.18

Shear force and bending moment diagrams for the beam of Ex. 3.10.

Now considering the moment equilibrium about X_1 of the length AX_1 of the beam in Fig. 3.18(b)

$$M_{AB} - \frac{M_0}{L}x = 0$$

or

$$M_{AB} = \frac{M_0}{L}x \tag{ii}$$

M_{AB} therefore increases linearly from zero at A ($x = 0$) to $+3M_0/4$ at B($x = 3L/4$). From Fig. 3.18(c) and taking moments about X_2 we have

$$M_{BC} + \frac{M_0}{L}(L - x) = 0$$

or

$$M_{BC} = \frac{M_0}{L}(x - L) \qquad \text{(iii)}$$

M_{BC} therefore decreases linearly from $-M_0/4$ at B($x = 3L/4$) to zero at C($x = L$); the complete distribution of bending moment is shown in Fig. 3.18(e). The deflected form of the beam is shown in Fig. 3.18(f) where a point of contraflexure occurs at B, the section at which the bending moment changes sign.

In this example, as in Ex. 3.9, the exact form of the deflected shape cannot be deduced from the bending moment diagram without analysis. However, using the method of singularities described in Section 13.2, it may be shown that the deflection at B is negative and that the slope of the beam at C is positive, giving the displaced shape shown in Fig. 3.18(f).

EXAMPLE 3.11

Construct shear force and bending moment diagrams for the truss shown in Fig.3.19 (a).

The support at E is a roller support so that only a vertical reaction, $R_{E,V}$, can occur there. Considering the horizontal equilibrium of the truss

$$R_{A,H} - 1 = 0$$

so that

$$R_{A,H} = 1 \text{ kN}$$

Now taking moments about E

$$R_{A,V} \times 6 - 5 \times 4 - 5 \times 3 - 10 \times 2 + 1 \times 1 = 0$$

which gives

$$R_{A,V} = 9 \text{ kN}$$

The vertical equilibrium of the truss gives

$$R_{E,V} + R_{A,V} - 5 - 5 - 10 = 0$$

from which

$$R_{E,V} = 11 \text{ kN}$$

With regard to vertical forces there are loading discontinuities at B, C and D; the horizontal load at F will not contribute to the shear force at any section of the truss. Initially, therefore, we consider a length, x, of the truss as shown in Fig. 3.19(b) and insert a positive shear force, S_{AB}, and a positive bending moment, M_{AB}, at the section X_1. Then, for vertical equilibrium of the length of truss

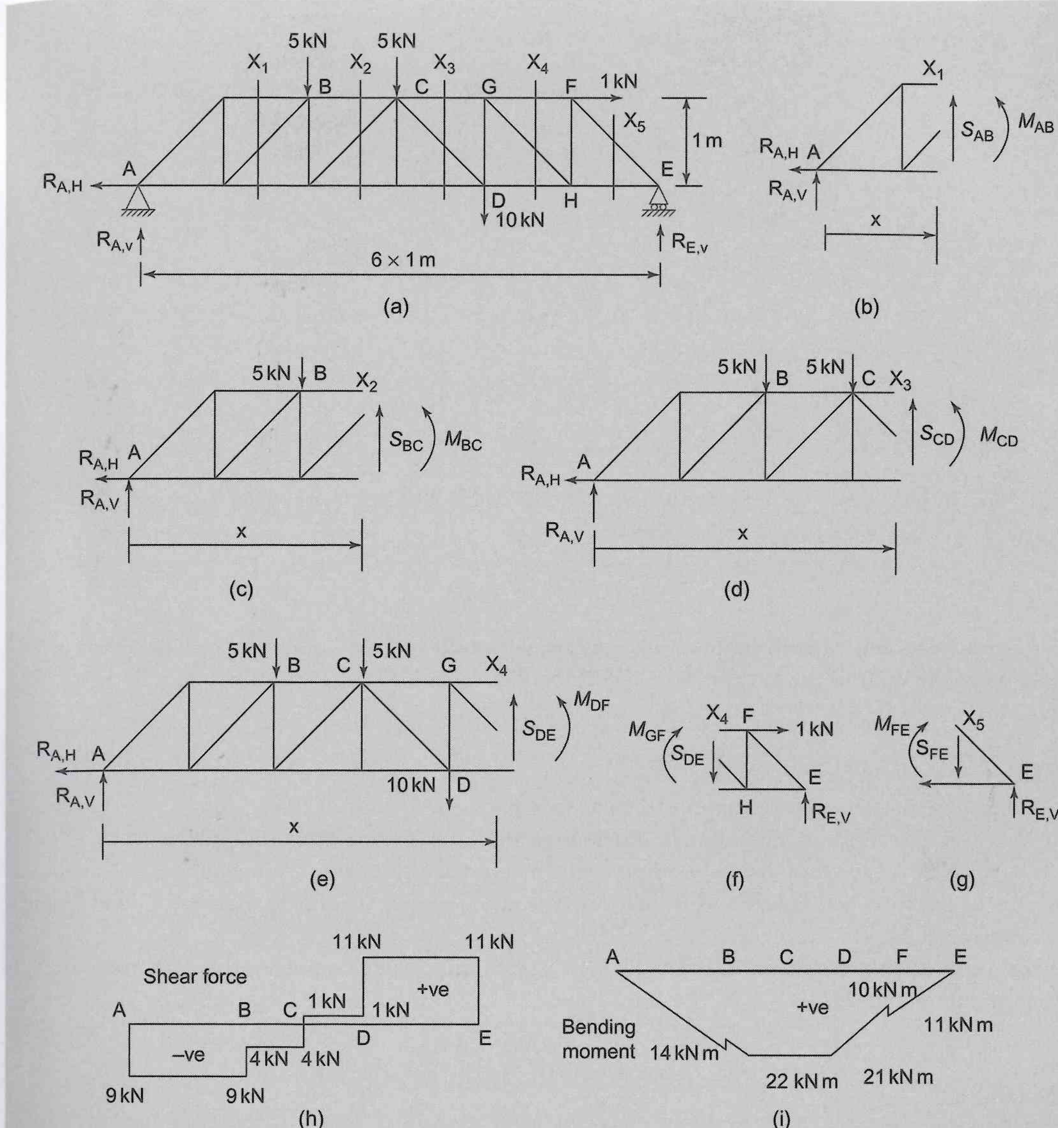


FIGURE 3.19

Shear force and bending moment diagrams for the truss of Ex. 3.11.

$$S_{AB} + R_{A,V} = 0$$

or

$$S_{AB} + 9 = 0$$

so that

$$S_{AB} = -9 \text{ kN}$$

Similarly, from Fig.3.19 (c)

$$S_{BC} + R_{A,V} - 5 = 0$$

which gives

$$S_{BC} = -4 \text{ kN}$$

Then, from Fig. 3.19(d)

$$S_{CD} + R_{A,V} - 5 - 5 = 0$$

from which

$$S_{CD} = +1 \text{ kN}$$

and from Fig. 3.19(e)

$$S_{DE} + R_{A,V} - 5 - 5 - 10 = 0$$

which gives

$$S_{DE} = +11 \text{ kN}$$

Alternatively, and slightly simpler, we could have considered the equilibrium of the portion of the truss to the right of the section X_4 as in Fig. 3.19(f). Then

$$S_{DE} - R_{E,V} = 0$$

which gives $S_{DE} = +11$ kN as before.

The complete shear force diagram is then as shown in Fig. 3.19(h).

With regard to the bending moment distribution there are loading discontinuities at B, C, D and also at F which is caused by the application of the horizontal 1 kN load. We must therefore consider sections of the truss between A and B, between B and C, between C and D, between D and F and between F and E.

Now considering the length of truss in Fig. 3.19(b) and taking moments about the section X_1 (thereby eliminating S_{AB})

$$M_{AB} - R_{A,V}x = 0$$

from which

$$M_{AB} = 9x \qquad \text{(i)}$$

Eq. (i) shows that M_{AB} varies linearly from zero at A to $9 \times 2 = +18$ kNm at B.

Similarly, from Fig. 3.19(c) and taking moments about the section X_2

$$M_{BC} - R_{A,V}x + 5(x - 2) = 0$$

so that

$$M_{BC} = 10 + 4x \qquad \text{(ii)}$$

Therefore, from Eq. (ii), M_{BC} varies linearly from +18 kNm at B($x = 2$ m) to +22 kNm at C($x = 3$ m).

Now, from Fig. 3.19(d) and taking moments about the section X_3

$$M_{CD} - R_{A,V}x + 5(x - 2) + 5(x - 3) = 0$$

which gives

$$M_{CD} = 25 - x \qquad \text{(iii)}$$

Eq. (iii) shows that M_{CD} varies linearly from +22 kNm at C($x = 3$ m) to +21 kNm at D($x = 4$ m).

The bending moment distribution in DF may be found by considering the equilibrium of the portion of the frame to the left of X_4 as shown in Fig. 3.19(e). Then, taking moments about X_4

$$M_{DF} - R_{A,V}x + 5(x - 2) + 5(x - 3) + 10(x - 4) = 0$$

from which

$$M_{DF} = 65 - 11x \qquad \text{(iv)}$$

Therefore, M_{DF} varies linearly from +21 kNm at D($x = 4$ m) to +10 kNm at F($x = 5$ m).

Now considering the length of truss to the right of the section X_5 and taking moments about the section X_5

$$M_{FE} - R_{E,V}(6 - x) = 0$$

which gives

$$M_{FE} = 66 - 11x \qquad \text{(v)}$$

so that M_{FE} varies linearly from +11 kNm at F ($x = 5$ m) to zero at E ($x = 6$ m); the complete bending moment distribution is then as shown in Fig. 3.19(i).

The discontinuity at F is due to the moment at F produced by the horizontal load at F which, together with the horizontal support reaction, $R_{A,H}$, may be regarded as forming a couple of magnitude $1 \times 1 = 1$ kNm acting at the truss section at F. (from the concept of the transmissibility of a force, $R_{A,H}$ may be regarded as acting at any point in its line of action). The situation is then similar to that in Ex. 3.10 where a point moment, M_0, is applied to the beam at B.

3.5 Load, shear force and bending moment relationships

It is clear from Exs 3.5–3.10 that load, shear force and bending moment are related. Thus, for example, uniformly distributed loads produce linearly varying shear forces and maximum values of bending moment coincide with zero shear force. We shall now examine these relationships mathematically.

The length of beam shown in Fig. 3.20(a) carries a general system of loading comprising concentrated loads and a distributed load $w(x)$. An elemental length δx of the beam is subjected to the force and moment system shown in Fig. 3.20(b); since δx is very small the distributed load may be regarded as constant over the length δx. For vertical equilibrium of the element

$$S + w(x)\delta x - (S + \delta S) = 0$$

so that

$$+w(x)\delta x - \delta S = 0$$

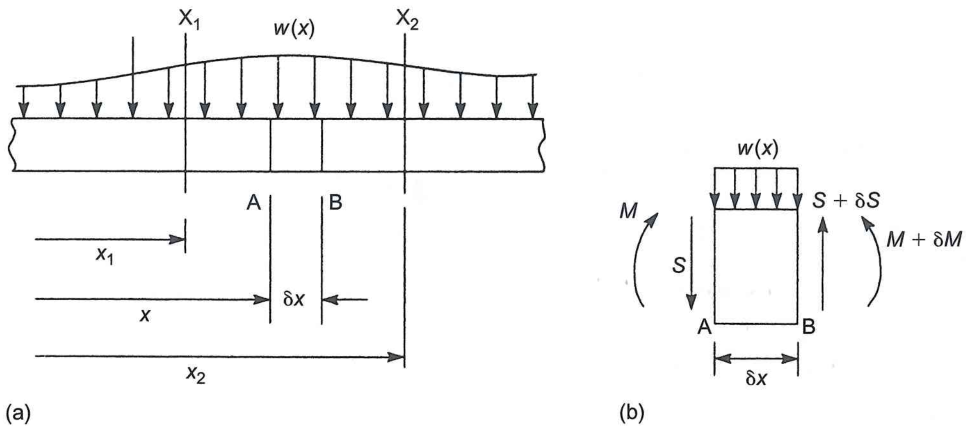


FIGURE 3.20

Load, shear force and bending moment relationships.

Thus, in the limit as $\delta x \to 0$

$$\frac{dS}{dx} = +w(x) \tag{3.1}$$

From Eq. (3.1) we see that the rate of change of shear force at a section of a beam, in other words the gradient of the shear force diagram, is equal to the value of the load intensity at that section. In Fig. 3.13(c), for example, the shear force changes linearly from $-wL$ at A to zero at B so that the gradient of the shear force diagram at any section of the beam is $+wL/L = +w$ where w is the load intensity. Equation (3.1) also applies at beam sections subjected to concentrated loads. In Fig. 3.14(a) the load intensity at B, theoretically, is infinite, as is the gradient of the shear force diagram at B (Fig. 3.14 (d)). In practice the shear force diagram would have a finite gradient at this section as illustrated in Fig. 3.15.

Now integrating Eq. (3.1) with respect to x we obtain

$$S = +\int w(x)\,dx + C_1 \tag{3.2}$$

in which C_1 is a constant of integration which may be determined in a particular case from the loading boundary conditions.

If, for example, $w(x)$ is a uniformly distributed load of intensity w, i.e., it is not a function of x, Eq. (3.2) becomes

$$S = +wx + C_1$$

which is the equation of a straight line of gradient $+w$ as demonstrated for the cantilever beam of Fig. 3.13 in the previous paragraph. Furthermore, for this particular example, $S = 0$ at $x = L$ so that $C_1 = -wL$ and $S = -w(L - x)$ as before.

In the case of a beam carrying only concentrated loads then, in the bays between the loads, $w(x) = 0$ and Eq. (3.2) reduces to

$$S = C_1$$

Suppose now that Eq. (3.1) is integrated over the length of beam between the sections X_1 and X_2. Then

$$\int_{x_1}^{x_2} \frac{dS}{dx}\,dx = +\int_{x_1}^{x_2} w(x)\,dx$$

which gives

$$S_2 - S_1 = \int_{x1}^{x_2} w(x)\,dx \tag{3.3}$$

where S_1 and S_2 are the shear forces at the sections X_1 and X_2 respectively. Equation (3.3) shows that the *change* in shear force between two sections of a beam is equal to the area under the load distribution curve over that length of beam.

The argument may be applied to the case of a concentrated load W which may be regarded as a uniformly distributed load acting over an extremely small elemental length of beam, say δx. The area under the load distribution curve would then be $w\delta x$ $(= W)$ and the change in shear force from the section x to the section $x + \delta x$ would be $+W$. In other words, the change in shear force from a section immediately to the left of a concentrated load to a section immediately to the right is equal to the value of the load, as noted in Ex. 3.7.

Now consider the rotational equilibrium of the element δx in Fig. 3.20(b) about B. Thus

$$M - S\delta x - w(x)\delta x\frac{\delta x}{2} - (M + \delta M) = 0$$

The term involving the square of δx is a second-order term and may be neglected. Hence

$$-S\delta x - \delta M = 0$$

or, in the limit as $\delta x \to 0$

$$\frac{dM}{dx} = -S \tag{3.4}$$

Equation (3.4) establishes for the general case what may be observed in particular in the shear force and bending moment diagrams of Exs 3.5–3.10, i.e. the gradient of the bending moment diagram at a beam section is equal to minus the value of the shear force at that section. For example, in Fig. 3.17(e) the bending moment in AB is a mathematical maximum at the section where the shear force is zero.

Integrating Eq. (3.4) with respect to x we have

$$M = -\int S\,dx + C_2 \tag{3.5}$$

in which C_2 is a constant of integration. Substituting for S in Eq. (3.5) from Eq. (3.2) gives

$$M = -\int \left[+ \int w(x)\,dx + C_1 \right] dx + C_2$$

or

$$M = -\int\int w(x)\,dx + C_1 x + C_2 \tag{3.6}$$

If $w(x)$ is a uniformly distributed load of intensity w, Eq. (3.6) becomes

$$M = -w\frac{x^2}{2} - C_1x + C_2$$

which shows that the equation of the bending moment diagram on a length of beam carrying a uniformly distributed load is parabolic.

In the case of a beam carrying concentrated loads only, then, between the loads, $w(x) = 0$ and Eq. (3.6) reduces to

$$M = -C_1x + C_2$$

which shows that the bending moment varies linearly between the loads and has a gradient $-C_1$.

The constants C_1 and C_2 in Eq. (3.6) may be found, for a given beam, from the loading boundary conditions. Thus, for the cantilever beam of Fig. 3.13, we have already shown that $C_1 = -wL$ so that $M = -wx^2/2 + wLx + C_2$. Also, when $x = L$, $M = 0$ which gives $C_2 = -wL^2/2$ and hence $M = -wx^2/2 + wLx - wL^2/2$ as before.

Now integrating Eq. (3.4) over the length of beam between the sections X_1 and X_2 (Fig. 3.20(a))

$$\int_{x_1}^{x_2} \frac{dM}{dx}dx = -\int_{x_1}^{x_2} S\,dx$$

which gives

$$M_2 - M_1 = -\int_{x_1}^{x_2} S\,dx \tag{3.7}$$

where M_1 and M_2 are the bending moments at the sections X_1 and X_2, respectively. Equation (3.7) shows that the *change* in bending moment between two sections of a beam is equal to minus the area of the shear force diagram between those sections. Again, using the cantilever beam of Fig. 3.13 as an example, we see that the change in bending moment from A to B is $wL^2/2$ and that the area of the shear force diagram between A and B is $-wL^2/2$.

Finally, from Eqs (3.1) and (3.4)

$$\frac{d^2M}{dx^2} = -\frac{dS}{dx} = -w(x) \tag{3.8}$$

The relationships established above may be used to construct shear force and bending moment diagrams for some beams more readily than when the methods illustrated in Exs 3.5–3.10 are employed. In addition they may be used to provide simpler solutions in some beam problems.

EXAMPLE 3.12

Construct shear force and bending moment diagrams for the beam shown in Fig. 3.21(a).

Initially the support reactions are calculated using the methods described in Section 2.5. Then, for moment equilibrium of the beam about E

$$R_A \times 4 - 2 \times 3 - 5 \times 2 - 4 \times 1 \times 0.5 = 0$$

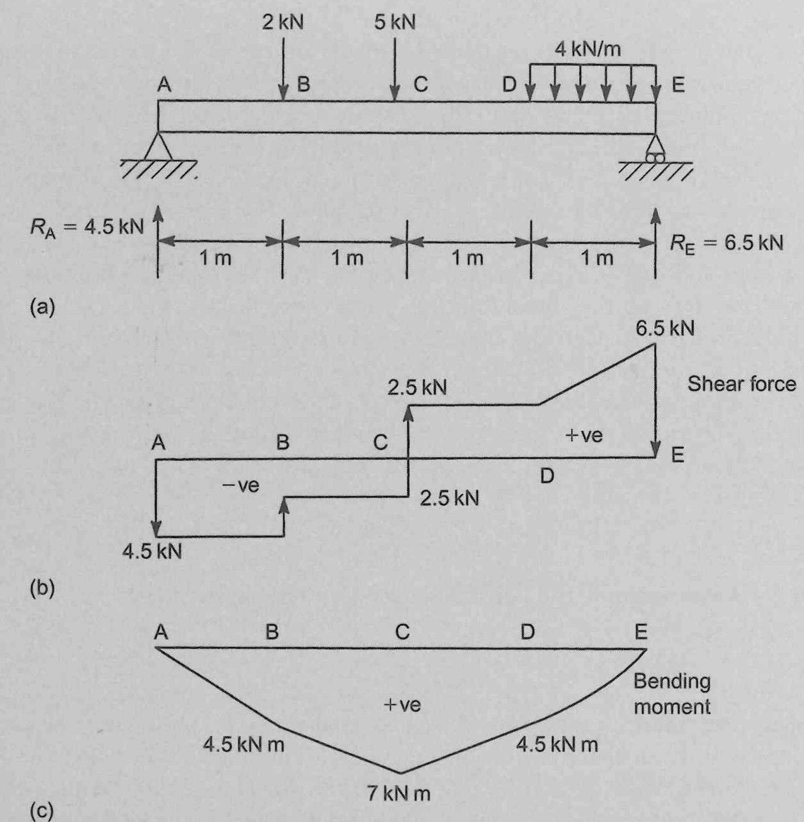


FIGURE 3.21

Shear force and bending moment diagrams for the beam of Ex. 3.12.

from which

$$R_A = 4.5 \text{ kN}$$

Now considering the vertical equilibrium of the beam

$$R_E + R_A - 2 - 5 - 4 \times 1 = 0$$

so that

$$R_E = 6.5 \text{ kN}$$

In constructing the shear force diagram we can make use of the facts that, as established above, the shear force is constant over unloaded bays of the beam, varies linearly when the loading is uniformly distributed and changes positively as a vertically downward concentrated load is crossed in the positive x direction by the value of the load. Thus in Fig. 3.21(b) the shear force increases negatively by 4.5 kN as we move from the left of A to the right of A, is constant between A and B, changes positively by 2 kN as we move from the left of B to the right of B, and so on. Note that between D and E the shear force changes linearly from +2.5 kN at D to +6.5 kN at a section immediately to the left of E, in other words it changes by +4 kN, the total value of the downward-acting uniformly distributed load.

The bending moment diagram may also be constructed using the above relationships, namely, the bending moment varies linearly over unloaded lengths of beam and parabolically over lengths of beam carrying a uniformly distributed load. Also, the change in bending moment between two sections of a beam is equal to minus the area of the shear force diagram between those sections. Thus in Fig. 3.21(a) we know that the bending moment at the pinned support at A is zero and that it varies linearly in the bay AB. The bending moment at B is then equal to minus the area of the shear force diagram between A and B, i.e. $-(-4.5\times 1)=$ 4.5 kN m. This represents, in fact, the change in bending moment from the zero value at A to the value at B. At C the area of the shear force diagram to the right or left of C is 7 kN m (note that the bending moment at E is also zero), and so on. In the bay DE the shape of the parabolic curve representing the distribution of bending moment over the length of the uniformly distributed load may be found using part of Eq. (3.8), i.e.

$$\frac{d^2M}{dx^2}=-w(x)$$

For a vertically downward uniformly distributed load this expression becomes

$$\frac{d^2M}{dx^2}=-w$$

which from mathematical theory shows that the curve representing the variation in bending moment is convex in the positive direction of bending moment. This may be observed in the bending moment diagrams in Fig. 3.13(d), 3.16(d) and 3.17(e). In this example the bending moment diagram for the complete beam is shown in Fig. 3.21(c) and is again drawn on the tension side of the beam.

EXAMPLE 3.13

A precast concrete beam of length L is to be lifted from the casting bed and transported so that the maximum bending moment is as small as possible. If the beam is lifted by two slings placed symmetrically, show that each sling should be $0.21L$ from the adjacent end.

The external load on the beam is comprised solely of its own weight, which is uniformly distributed along its length. The problem is therefore resolved into that of a simply supported beam carrying a uniformly distributed load in which the supports are positioned at some distance a from each end (Fig. 3.22(a)).

The shear force and bending moment diagrams may be constructed in terms of a using the methods described above and would take the forms shown in Fig. 3.22(b) and (c). Examination of the bending moment diagram shows that there are two possible positions for the maximum bending moment. First at B and C where the bending moment is hogging and has equal values from symmetry; second at the mid-span point where the bending moment has a turning value and is sagging if the supports at B and C are spaced a sufficient distance apart. Suppose that B and C are positioned

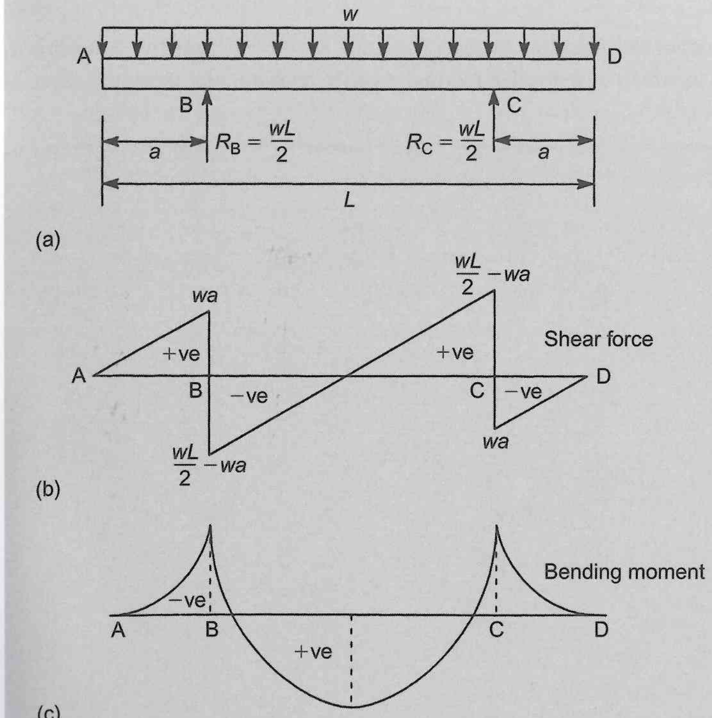


FIGURE 3.22
Determination of optimum position for supports in the precast concrete beam of Ex. 3.13.

such that the value of the hogging bending moment at B and C is numerically equal to the sagging bending moment at the mid-span point. If now B and C are moved further apart the mid-span moment will increase while the moment at B and C decreases. Conversely, if B and C are brought closer together, the hogging moment at B and C increases while the mid-span moment decreases. It follows that the maximum bending moment will be as small as possible when the hogging moment at B and C is numerically equal to the sagging moment at mid-span.

The solution will be simplified if use is made of the relationship in Eq. (3.7). Thus, when the supports are in the optimum position, the change in bending moment from A to B (negative) is equal to minus half the change in the bending moment from B to the mid-span point (positive). It follows that the area of the shear force diagram between A and B is equal to minus half of that between B and the mid-span point. Then

$$+\frac{1}{2}awa=-\frac{1}{2}\left[-\frac{1}{2}\left(\frac{L}{2}-a\right)w\left(\frac{L}{2}-a\right)\right]$$

which reduces to

$$a^2+La-\frac{L^2}{4}=0$$

the solution of which gives

$$a=0.21L\quad\text{(the negative solution has no practical significance)}$$

3.6 Torsion

The distribution of torque along a structural member may be obtained by considering the equilibrium in free body diagrams of lengths of member in a similar manner to that used for the determination of shear force distributions in Exs 3.5–3.10.

EXAMPLE 3.14

Construct a torsion diagram for the beam shown in Fig. 3.23(a).

There is a loading discontinuity at B so that we must consider the torque at separate sections X_1 and X_2 in AB and BC, respectively. Thus, in the free body diagrams shown in Fig. 3.23(b) and (c) we insert positive internal torques.

From Fig. 3.23(b)

$$T_{AB} - 10 + 8 = 0$$

so that

$$T_{AB} = +2 \text{ kN m}$$

From Fig. 3.23(c)

$$T_{BC} + 8 = 0$$

from which

$$T_{BC} = -8 \text{ kN m}$$

The complete torsion diagram is shown in Fig. 3.23(d).

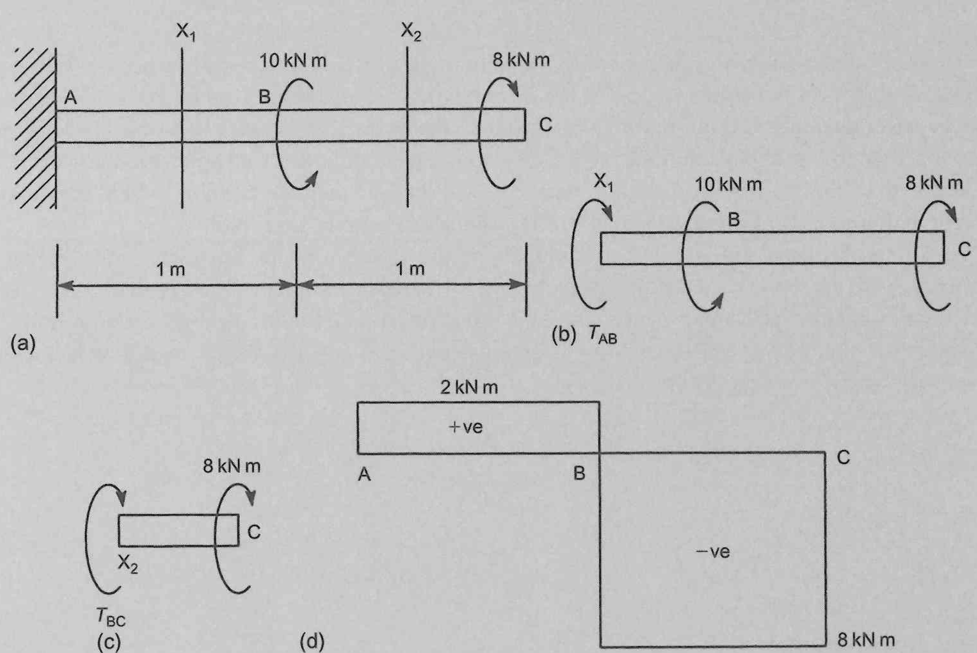


FIGURE 3.23

EXAMPLE 3.15

The structural member ABC shown in Fig. 3.24 carries a distributed torque of 2 kNm/m together with a concentrated torque of 10 kN m at mid-span. The supports at A and C prevent rotation of the member in planes perpendicular to its axis. Construct a torsion diagram for the member and determine the maximum value of torque.

From the rotational equilibrium of the member about its longitudinal axis and its symmetry about the mid-span section at B, we see that the reactive torques T_A and T_C are each -9 kN m, i.e. clockwise when viewed in the direction CBA. In general, as we shall see in Chapter 11, reaction torques at supports form a statically indeterminate system.

In this particular problem there is a loading discontinuity at B so that we must consider the internal torques at two arbitrary sections X_1 and X_2 as shown in Fig. 3.25(a).

From the free body diagram in Fig. 3.25(b)

$$T_{AB} + 2x - 9 = 0$$

which gives

$$T_{AB} = 9 - 2x \qquad \text{(i)}$$

From Eq. (i) we see that T_{AB} varies linearly from +9 kNm at A ($x = 0$) to +5 kNm at a section immediately to the left of B ($x = 2$ m). Furthermore, from Fig. 3.25(c)

$$T_{BC} - 2(4 - x) + 9 = 0$$

so that

$$T_{BC} = -2x - 1 \qquad \text{(ii)}$$

from which we see that T_{BC} varies linearly from -5 kN m at a section immediately to the right of B ($x = 2$ m) to -9 kNm at C ($x = 4$ m). The resulting torsion diagram is shown in Fig. 3.25(d).

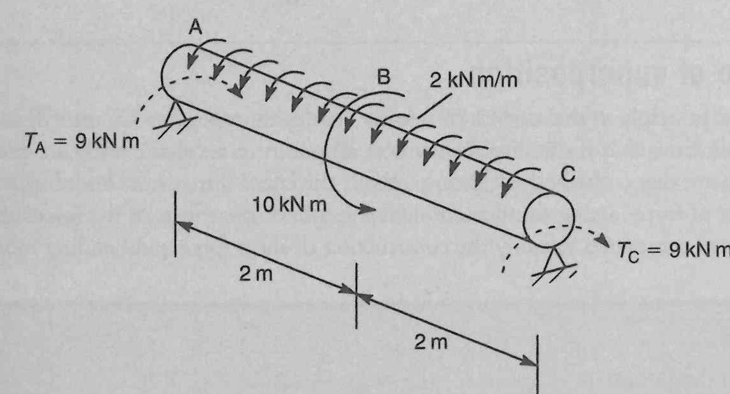


FIGURE 3.24
Beam of Ex. 3.15.

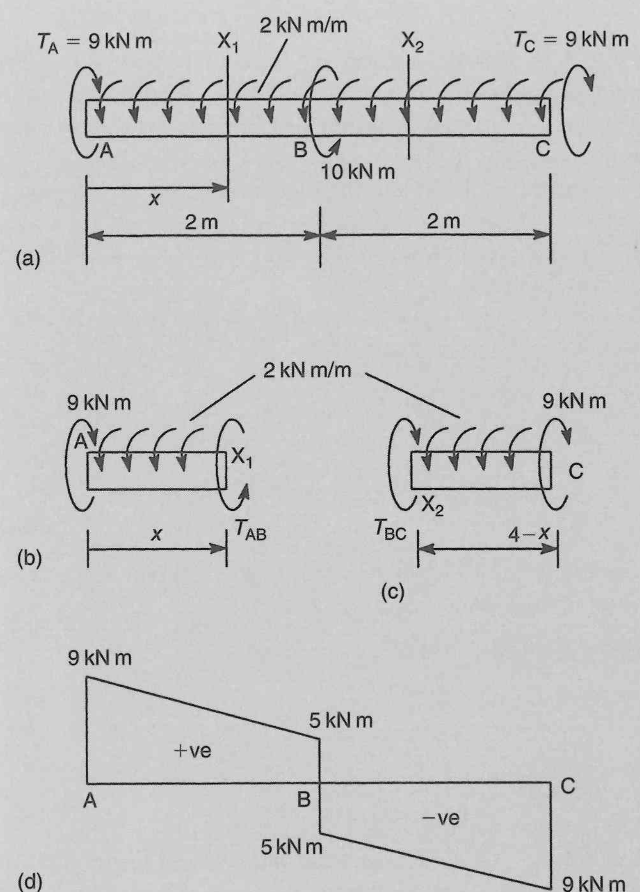


FIGURE 3.25

Torsion diagram for the beam of Ex. 3.15.

3.7 Principle of superposition

An extremely useful principle in the analysis of linearly elastic structures (see Chapter 8) is that of superposition. The principle states that if the displacements at all points in an elastic body are proportional to the forces producing them, that is the body is linearly elastic, the effect (i.e. stresses and displacements) on such a body of a number of forces acting simultaneously is the sum of the effects of the forces applied separately.

This principle can sometimes simplify the construction of shear force and bending moment diagrams.

EXAMPLE 3.16

Construct the bending moment diagram for the beam shown in Fig. 3.26(a).

Figures 3.26(b), (c) and (d) show the bending moment diagrams for the cantilever when each of the three loading systems acts separately. The bending moment diagram for the beam when the loads act simultaneously is obtained by adding the ordinates of the separate diagrams and is shown in

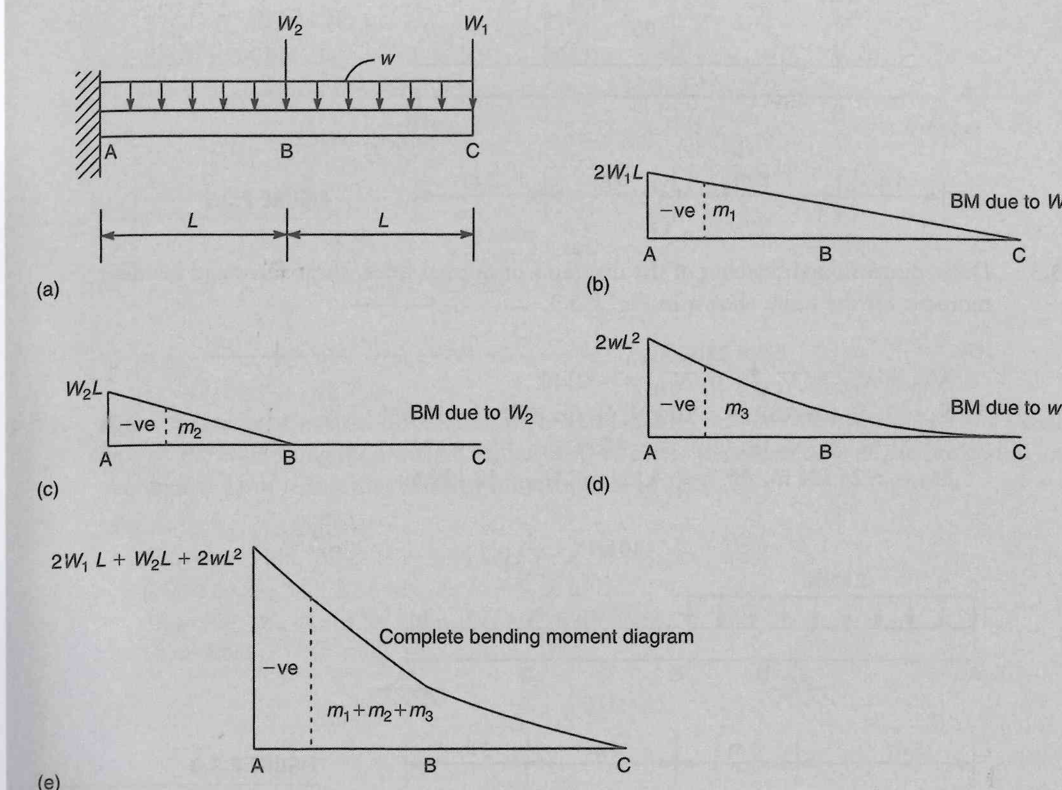


FIGURE 3.26

Bending moment (BM) diagram using the principle of superposition.

PROBLEMS

P.3.1 A transmitting mast of height 40 m and weight 4.5 kN/m length is stayed by three groups of four cables attached to the mast at heights of 15, 25 and 35 m. If each cable is anchored to the ground at a distance of 20 m from the base of the mast and tensioned to a force of 15 kN, draw a diagram of the compressive force in the mast.

Ans. Max. force = 314.9 kN.

P.3.2 Construct the normal force, shear force and bending moment diagrams for the beam shown in Fig. P.3.2.

Ans.

N_{AB} = 9.2 kN, N_{BC} = 9.2 kN, N_{CD} = 5.7 kN, N_{DE} = 0.
S_{AB} = −6.9 kN, S_{BC} = −3.9 kN, S_{CD} = +2.2 kN, S_{DE} = +7.9 kN.
M_B = 27.6 kNm, M_C = 51 kN m, M_D = 40 kN m.

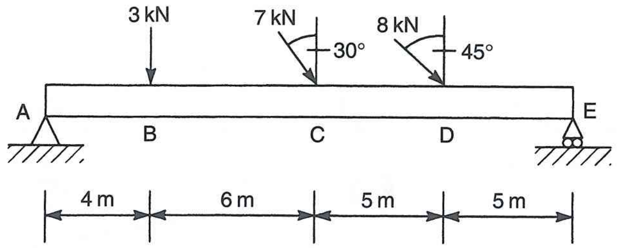


FIGURE P.3.2

P.3.3 Draw dimensioned sketches of the diagrams of normal force, shear force and bending moment for the beam shown in Fig. P.3.3.

Ans.

$N_{AB} = N_{BC} = N_{CD} = 0$, $N_{DE} = -6$ kN.
$S_A = 0$, S_B (in AB) $= +10$ kN, S_B (in BC) $= -10$ kN.
$S_C = -4$ kN, S_D (in CD) $= -4$ kN, $S_{DE} = +4$ kN.
$M_B = -25$ kN m, $M_C = -4$ kN m, $M_D = 12$ kN m.

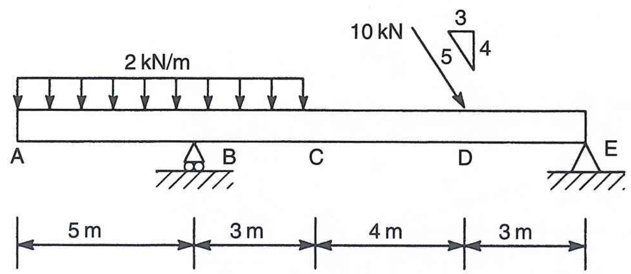


FIGURE P.3.3

P.3.4 Draw normal force, shear force and bending moment diagrams for the beam ABC shown in Fig. P.3.4. Insert all the principal values.

Ans.

$N_{AB} = +20$ kN, $N_{BC} = +10$ kN.
$S_A = +56.6$ kN, S_B (AB) $= +39.1$ kN, S_B (BC) $= +24.1$ kN, $S_C = +15$ kN.
$M_A = -181.0$ kNm, M_B (AB) $= -61.4$ kNm, M_B (BC) $= -43.4$ kNm,
$M_C = -18.0$ kNm.

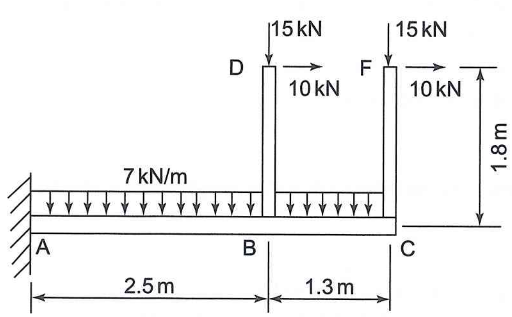


FIGURE P.3.4

P.3.5 Draw diagrams of normal force, shear force and bending moment for the cranked cantilever

Ans.

$N_{AB} = 0$, $N_{BC} = +2$ kN, $N_{CD} = 0$.
$S_{AB} = +14$ kN, $S_{BC} = +3.46$ kN, S_C (CD) $= +4$ kN, $S_D = 0$.
$M_A = -20$ kNm, $M_B = -6$ kNm, $M_C = -2$ kNm, $M_D = 0$.

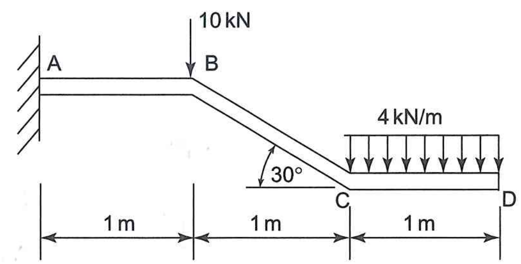


FIGURE P.3.5

P.3.6 Draw diagrams of normal force, shear force and bending moment for the beam ABCD shown in Fig. P.3.6 inserting all principal values. Also calculate the magnitude of the horizontal load required at D to reduce the bending moment at A to zero.

Ans.

$N_{DC} = -5$ kN, $N_{CBA} = -3.54$ kN,
$S_{DC} = 0$, $S_{CB} = +3.54$ kN, $S_A = +5.54$ kN.
$M_{DC} = 0$, $M_B = -3.54$ kNm, $M_A = -8.08$ kNm.
11.43 kN.

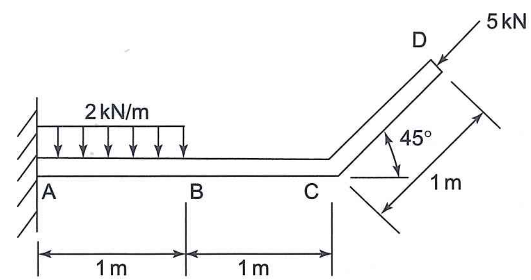


FIGURE P.3.6

P.3.7 Draw shear force and bending moment diagrams for the beam shown in Fig. P.3.7.

Ans.

$S_{AB} = -W$, $S_{BC} = 0$, $S_{CD} = +W$.
$M_B = M_C = WL/4$.

Note zero shear and constant bending moment in central span.

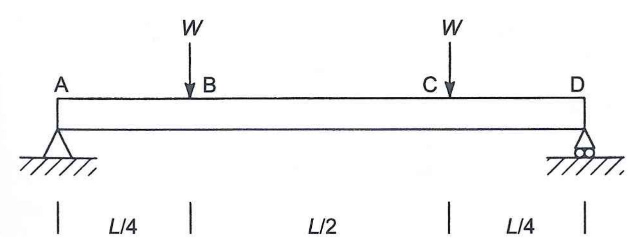


FIGURE P.3.7

P.3.8 The cantilever AB shown in Fig. P.3.8 carries a uniformly distributed load of 5 kN/m and a concentrated load of 15 kN at its free end. Construct the shear force and bending moment diagrams for the beam.

Ans.
$S_B = -15$ kN, $S_C = -65$ kN.
$M_B = 0$, $M_A = -400$ kN m.

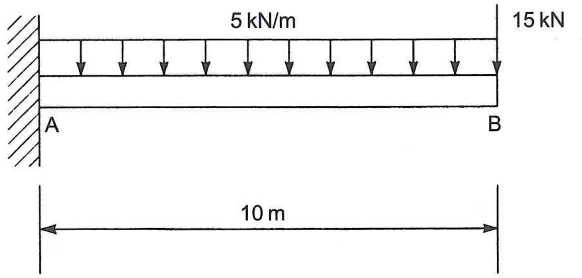


FIGURE P.3.8

P.3.9 Sketch the bending moment and shear force diagrams for the simply supported beam shown in Fig. P.3.9 and insert the principal values.

Ans.
S_B (in AB) = +5 kN, S_B (in BC) = −3.75 kN, S_C (in BC) = +6.25 kN.
$S_{CD} = -5$ kN, $M_B = -12.5$ kN m, $M_C = -25$ kN m.

Turning value of bending moment of −5.5 kN m in BC, 3.75 m from B.

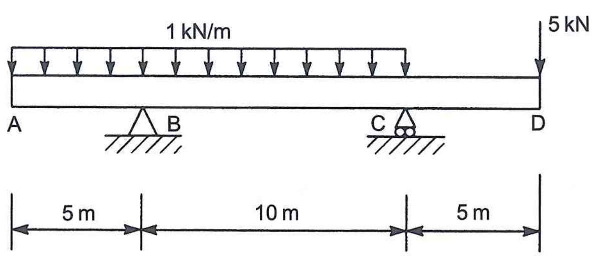
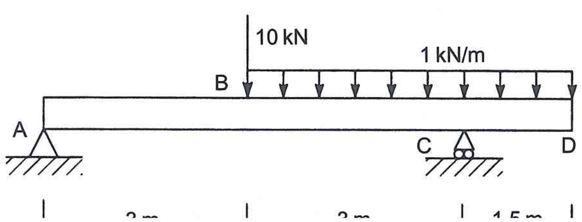


FIGURE P.3.9

P.3.10 Draw the shear force and bending moment diagrams for the beam shown in Fig. P.3.10 indicating the principal values.

Ans.
$S_{AB} = -5.6$ kN, S_B (in BC) = +4.4 kN, S_C (in BC) = +7.4 kN, S_C (in CD) = −1.5 kN.
$M_B = 16.8$ kNm, $M_C = -1.125$ kNm.



P.3.11 Find the value of w in the beam shown in Fig. P.3.11 for which the maximum sagging bending moment occurs at a point 10/3 m from the left-hand support and determine the value of this moment.

Ans. $w = 1.2$ kN/m, 6.7 kNm.

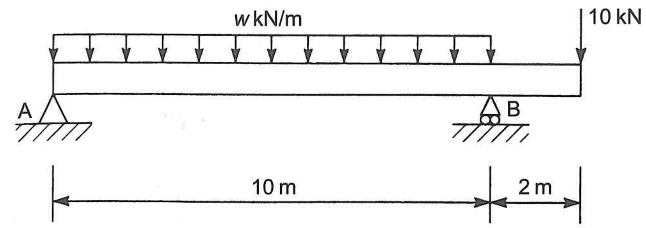


FIGURE P.3.11

P.3.12 Find the value of n for the beam shown in Fig. P.3.12 such that the maximum sagging bending moment occurs at $L/3$ from the right-hand support. Using this value of n determine the position of the point of contraflexure in the beam.

Ans. $n = 4/3$, $L/3$ from left-hand support.

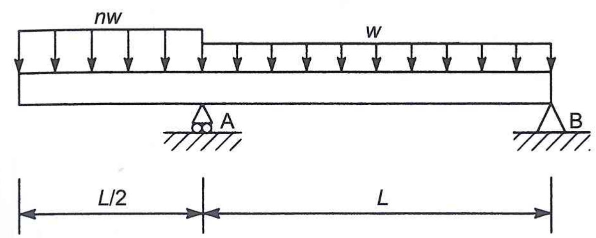


FIGURE P.3.12

P.3.13 Sketch the shear force and bending moment diagrams for the simply supported beam shown in Fig. P.3.13 and determine the positions of maximum bending moment and point of contraflexure. Calculate the value of the maximum moment.

Ans.
$S_A = -45$ kN, S_B (in AB) = +55 kN, $S_{BC} = -20$ kN.
$M_{max} = 202.5$ kN m at 9 m from A, $M_B = -100$ kN m.

Point of contraflexure is 18 m from A.

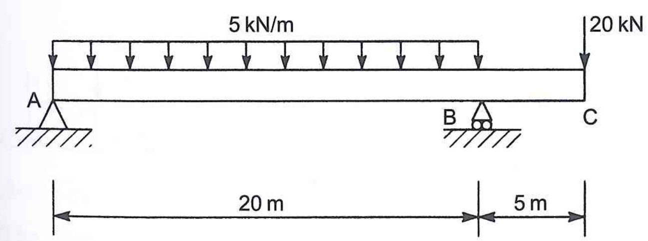


FIGURE P.3.13

P.3.14 Determine the position of maximum bending moment in a simply supported beam, 8 m span, which carries a load of 100 kN uniformly distributed over its complete length and, in from the left support. Calculate the value of maximum bending moment and the value of bending moment at mid-span.

Ans.
$M_{max} = 294$ kN m at 3.6 m from left-hand support.
M (mid-span) $= 289$ kN m.

P.3.15 A simply supported beam AB has a span of 6 m and carries a distributed load which varies linearly in intensity from zero at A to 2 kN/m at B. Sketch the shear force and bending moment diagrams for the beam and insert the principal values.

Ans.
$S_{AB} = -2 + x^2/6$, $S_A = -2$ kN, $S_B = +4$ kN.
$M_{AB} = 2x - x^3/18$, $M_{max} = 4.62$ kN m at 3.46 m from A.

P.3.16 A precast concrete beam of length L is to be lifted by a single sling and has one end resting on the ground. Show that the optimum position for the sling is 0.29 m from the nearest end.

P.3.17 Construct shear force and bending moment diagrams for the framework shown in Fig. P.3.17.

Ans.
$S_{AB} = -60$ kN, $S_{BC} = -10$ kN, $S_{CD} = +140$ kN.
$M_B = 480$ kN m, $M_C = 560$ kN m.

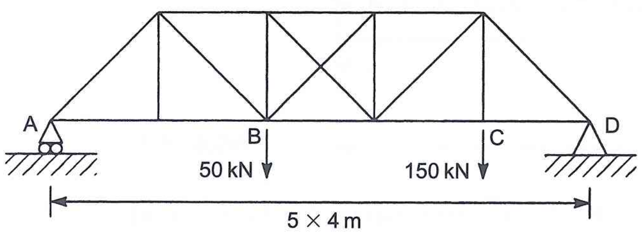


FIGURE P.3.17

P.3.18 Draw shear force and bending moment diagrams for the framework shown in Fig. P.3.18.

Ans.
$S_{AB} = +5$ kN, $S_{BC} = +15$ kN, $S_{CD} = +30$ kN, $S_{DE} = -12$ kN, $S_{EF} = -7$ kN,
$S_{FG} = -5$ kN, $S_{GH} = 0$.
$M_B = -10$ kN m, $M_C = -40$ kN m, $M_D = -100$ kN m, $M_E = -76$ kN m,
$M_F = -20$ kN m, $M_G = M_H = 0$.

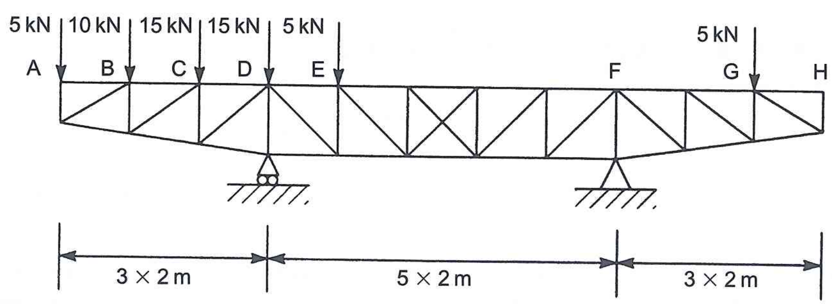


FIGURE P.3.18

P.3.19 Draw shear force and bending moment diagrams for the truss shown in Fig. P.3.19.

Ans.
$S_{AC} = -4$ kN, $S_{CD} = +1$ kN, $S_{DE} = +5$ kN.
$M_A = +1$ kNm, $M_C = +5$ kNm, $M_D = +3$ kNm, $M_E = -2$ kNm.

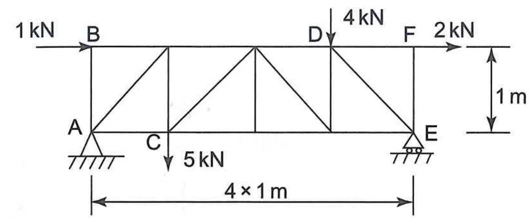


FIGURE P.3.19

P.3.20 The cranked cantilever ABC shown in Fig. P.3.20 carries a load of 3 kN at its free end. Draw shear force, bending moment and torsion diagrams for the complete beam.

Ans.
$S_{CB} = -3$ kN, $S_{BA} = -3$ kN
$M_C = 0$, M_B (in CB) $= -6$ kN m, M_B (in BA) $= 0$, $M_A = -9$ kNm.
$T_{CB} = 0$, $T_{BA} = 6$ kNm.

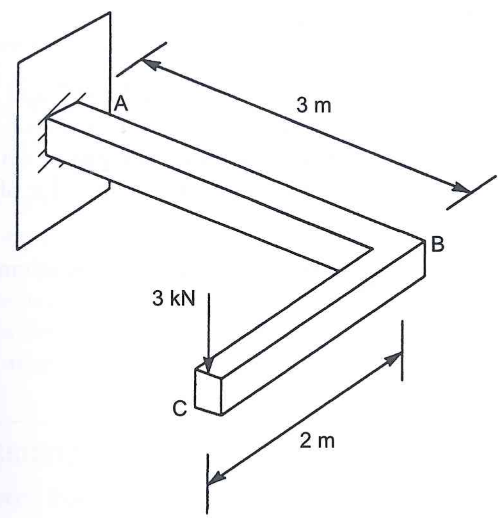


FIGURE P.3.20

P.3.21 Construct a torsion diagram for the beam shown in Fig. P.3.21.

Ans. $T_{CB} = -300$ N m, $T_{BA} = -400$ N m.

P.3.22 The beam ABC shown in Fig. P.3.22 carries a distributed torque of 1 Nm/mm over its outer half BC and a concentrated torque of 500 N m at B. Sketch the torsion diagram for the beam inserting the principal values.

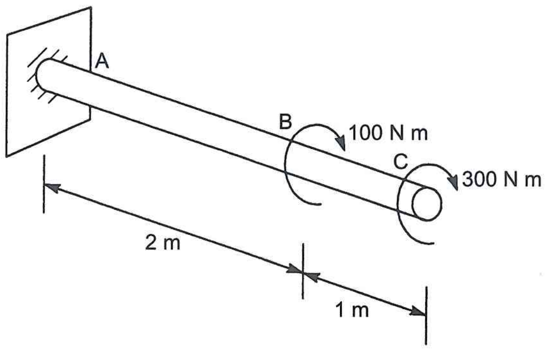


FIGURE P.3.21

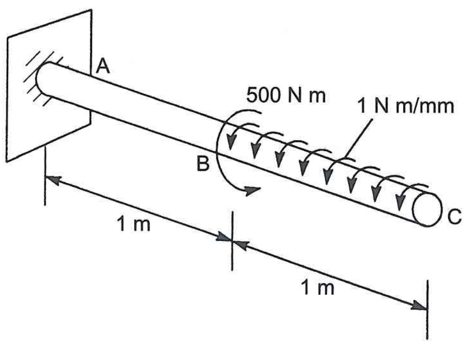


FIGURE P.3.22

P.3.23 The cylindrical bar ABCD shown in Fig. P.3.23 is supported symmetrically at B and C by supports that prevent rotation of the bar about its longitudinal axis. The bar carries a uniformly distributed torque of 2 N m/mm together with concentrated torques of 400 N m at each end. Draw the torsion diagram for the bar and determine the maximum value of torque.

Ans.

$T_{DC} = 400 + 2x$, $T_{CB} = 2x - 2000$, $T_{BA} = 2x - 4400$ (T in N m when x is in mm).
$T_{max} = 1400$ N m at C and B.

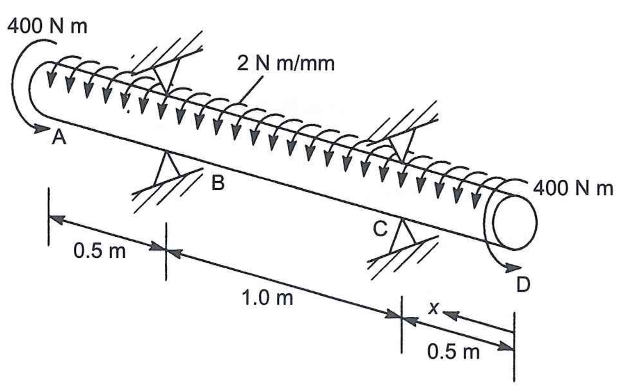


FIGURE P.3.23

Analysis of Pin-Jointed Trusses

In Chapter 1 we discussed various structural forms and saw that for moderately large spans, simple beams become uneconomical and may be replaced by trusses. These structures comprise members connected at their ends and are constructed in a variety of arrangements. In general, trusses are lighter, stronger and stiffer than solid beams of the same span; they do, however, take up more room and are more expensive to fabricate.

Initially in this chapter we shall discuss types of truss, their function and the idealization of a truss into a form amenable to analysis. Subsequently, we shall investigate the criterion which indicates the degree of their statical determinacy, examine the action of the members of a truss in supporting loads and, finally, examine methods of analysis of both plane and space trusses.

4.1 Types of truss

Generally the form selected for a truss depends upon the purpose for which it is required. Examples of different types of truss are shown in Fig. 4.1(a)–(f); some are named after the railway engineers who invented them.

For example, the Pratt, Howe, Warren and K trusses would be used to support bridge decks and large-span roofing systems (the Howe truss is no longer used for reasons we shall discuss in Section 4.5) whereas the Fink truss would be used to support gable-ended roofs. The Bowstring truss is somewhat of a special case in that if the upper chord members are arranged such that the joints lie on a parabola and the loads, all of equal magnitude, are applied at the upper joints, the internal members carry no load. This result derives from arch theory (Chapter 6) but is rarely of practical significance since, generally, the loads would be applied to the lower chord joints as in the case of the truss being used to support a bridge deck.

Frequently, plane trusses are connected together to form a three-dimensional structure. For example, in the overhead crane shown in Fig. 4.2, the tower would usually comprise four plane trusses joined together to form a 'box' while the jibs would be constructed by connecting three plane trusses together to form a triangular cross section.

4.2 Assumptions in truss analysis

It can be seen from Fig. 4.1 that plane trusses consist of a series of triangular units. The triangle, even when its members are connected together by hinges or pins as in Fig. 4.3(a), is an inherently stable structure, i.e. it will not collapse under any arrangement of loads applied in its own plane. On the other hand, the rectangular structure shown in Fig. 4.3(b) would be unstable if vertical loads were applied at the joints and would collapse under the loading system shown; in other words it is a mechanism.

Further properties of a pin-jointed triangular structure are that the forces in the members are purely axial and that it is statically determinate (see Section 4.4) so long as the structure is loaded and supported at the joints. The forces in the members can then be found using the equations of statical equilibrium

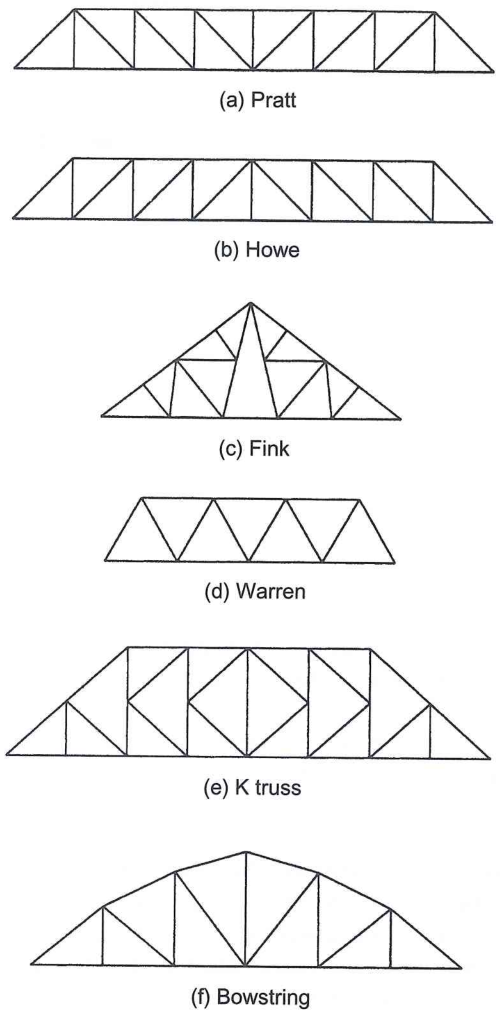


FIGURE 4.1

Types of plane truss.

(Eq. (2.10)). It follows that a truss comprising pin-jointed triangular units is also statically determinate if the above loading and support conditions are satisfied. In Section 4.4 we shall derive a simple test for determining whether or not a pin-jointed truss is statically determinate; this test, although applicable in most cases is not, as we shall see, foolproof.

The assumptions on which the analysis of trusses is based are as follows:

1. The members of the truss are connected at their ends by frictionless pins or hinges.
2. The truss is loaded and supported only at its joints.
3. The forces in the members of the truss are purely axial.

Assumptions (2) and (3) are interdependent since the application of a load at some point along a truss member would, in effect, convert the member into a simply supported beam and, as we have seen in Chapter 3, generate, in addition to axial loads, shear forces and bending moments; the truss would

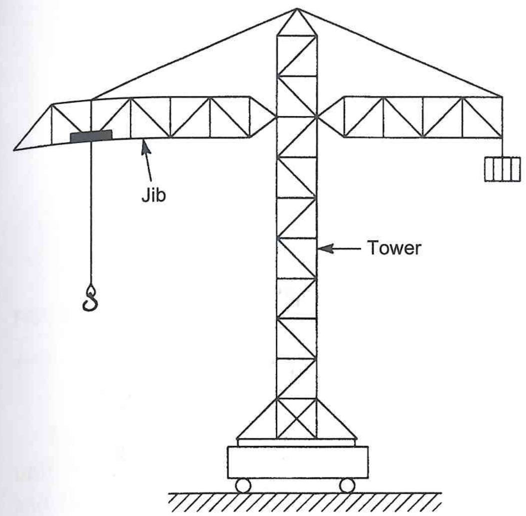


FIGURE 4.2

Overhead crane structure.

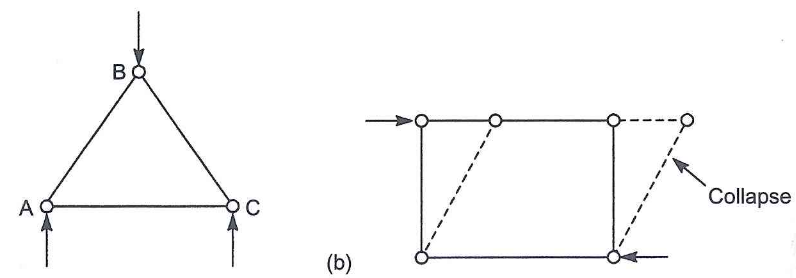


FIGURE 4.3

Basic unit of a truss.

4.3 Idealization of a truss

In practice trusses are not pin-jointed but are constructed, in the case of steel trusses, by bolting, riveting or welding the ends of the members to gusset plates as shown in Fig. 4.4. In a timber roof truss the members are connected using spiked plates driven into their vertical surfaces on each side of a joint. The joints in trusses are therefore semi-rigid and can transmit moments, unlike a frictionless pinned joint. Furthermore, if the loads are applied at points on a member away from its ends, that member behaves as a fixed or built-in beam with unknown moments and shear forces as well as axial loads at its ends. Such a truss would possess a high degree of statical indeterminacy and would require a computer-based analysis.

However, if such a truss is built up using the basic triangular unit and the loads and support points coincide with the member joints then, even assuming rigid joints, a computer-based analysis would show that the shear forces and bending moments in the members are extremely small compared to the axial forces which, themselves, would be very close in magnitude to those obtained from an analysis

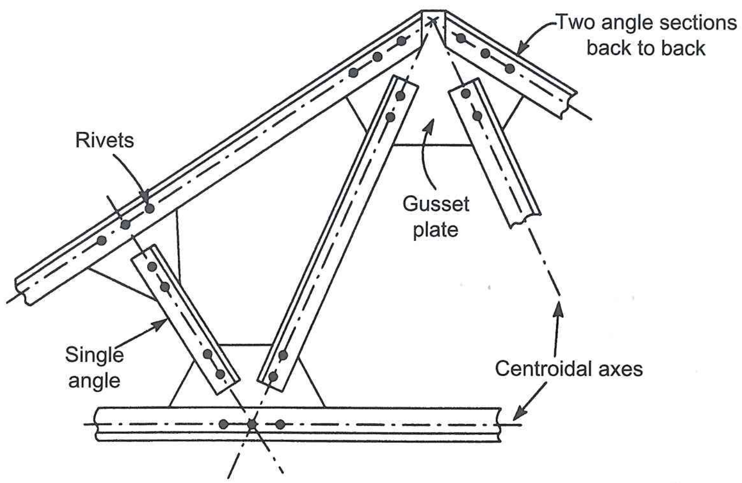


FIGURE 4.4
Actual truss construction.

A further condition in employing a pin-jointed idealization of an actual truss is that the centroidal axes of the members in the actual truss are concurrent, as shown in Fig. 4.4. We shall see in Section 9.2 that a load parallel to, but offset from, the centroidal axis of a member induces a bending moment in the cross-section of the member; this situation is minimized in an actual truss if the centroidal axes of all members meeting at a joint are concurrent.

4.4 Statical determinacy

It was stated in Section 4.2 that the basic triangular pin-jointed unit is statically determinate and the forces in the members are purely axial so long as the loads and support points coincide with the joints. The justification for this is as follows. Consider the joint B in the triangle in Fig. 4.3(a). The forces acting on the actual pin or hinge are the externally applied load and the axial forces in the members AB and BC; the system is shown in the free body diagram in Fig. 4.5. The internal axial forces in the members BA and BC, F_{BA} and F_{BC}, are drawn to show them pulling away from the joint B; this indicates that the members are in tension. Actually, we can see by inspection that both members will be in compression since their combined vertical components are required to equilibrate the applied vertical load. The assumption of tension, however, would only result in negative values in the calculation of F_{BA} and F_{BC} and is therefore a valid approach. In fact we shall adopt the method of initially assuming tension in all members of a truss when we consider methods of analysis, since a negative value for a member force will then always signify compression and will be in agreement with the sign convention adopted in Section 3.2.

Since the pin or hinge at the joint B is in equilibrium and the forces acting on the pin are coplanar, Eq. (2.10) apply. Therefore the sum of the components of all the forces acting on the pin in any two directions at right angles must be zero. The moment equation, $\sum M = 0$, is automatically satisfied since the pin cannot transmit a moment and the lines of action of all the forces acting on the pin must therefore be concurrent. For the joint B, we can write down two equations of force equilibrium which are sufficient to solve for the unknown member forces F_{BA} and F_{BC}. The same argument may then be applied to either joint A or C to solve for the remaining unknown internal force F_{AC} ($=F_{CA}$). We see

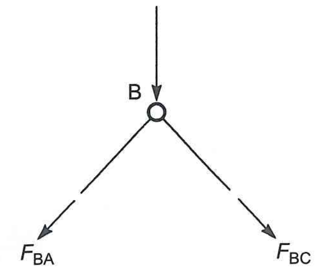


FIGURE 4.5
Joint equilibrium in a triangular structure.

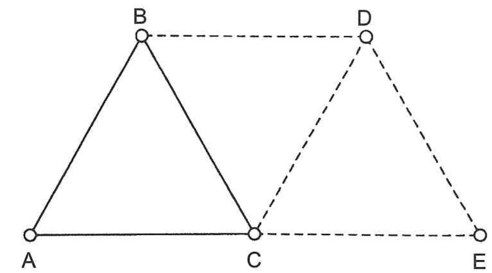


FIGURE 4.6
Construction of a Warren truss.

Now consider the construction of a simple pin-jointed truss. Initially we start with a single triangular unit ABC as shown in Fig. 4.6. A further triangle BCD is created by adding the *two* members BD and CD and the *single* joint D. The third triangle CDE is then formed by the addition of the *two* members CE and DE and the *single* joint E and so on for as many triangular units as required. Thus, after the initial triangle is formed, each additional triangle requires *two* members and a *single* joint. In other words the number of additional members is equal to twice the number of additional joints. This relationship may be expressed quantitatively as follows.

Suppose that m is the total number of members in a truss and j the total number of joints. Then, noting that initially there are three members and three joints, the above relationship may be written

$$m - 3 = 2(j - 3)$$

so that

$$m = 2j - 3 \tag{4.1}$$

If Eq. (4.1) is satisfied, the truss is constructed from a series of statically determinate triangles and the truss itself is statically determinate. Furthermore, if $m < 2j-3$ the structure is unstable (see Fig. 4.3(b)) or if $m > 2j-3$, the structure is statically indeterminate. Note that Eq. (4.1) applies only to the internal forces in a truss; the support system must also be statically determinate to enable the analysis to be carried out using simple statics.

EXAMPLE 4.1

Test the statical determinacy of the pin-jointed trusses shown in Fig. 4.7.

In Fig. 4.7(a) the truss has five members and four joints so that $m = 5$ and $j = 4$. Then

$$2j - 3 = 5 = m$$

and Eq. (4.1) is satisfied. The truss in Fig. 4.7(b) has an additional member so that $m = 6$ and $j = 4$. Therefore

$$m > 2j - 3$$

and the truss is statically indeterminate.

The truss in Fig. 4.7(c) comprises a series of triangular units which suggests that it is statically determinate. However, in this case, $m = 8$ and $j = 5$. Thus