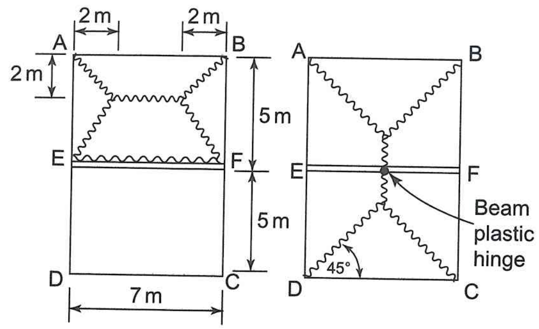


FIGURE P.19.8

.9 The reinforced concrete slab shown in Fig. P.19.9 is simply supported on all its outer edges and has an additional wall support internally along the line aa over which the slab is continuous. The ultimate moments of resistance for sagging bending are isotropic and of value m per unit width while the ultimate moments for hogging bending are also isotropic and of value 1.2 m per unit width. If the design ultimate load is 14 kN/m^2 over the whole slab area and the yield line pattern is as shown find the value of the moment parameter m.
Ans. 22.05 kNm/m.

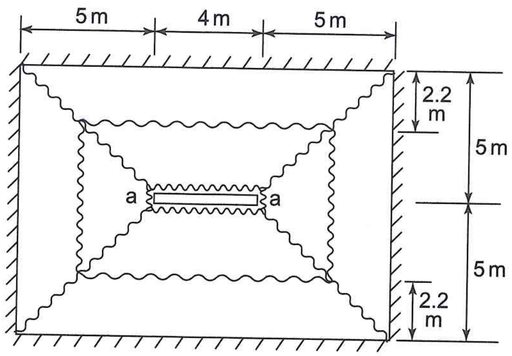


FIGURE P.19.9

CHAPTER

Influence Lines 20

The structures we have considered so far have been subjected to loading systems that were stationary, i. e. the loads remained in a fixed position in relation to the structure. In many practical situations, however, structures carry loads that vary continuously. For example, a building supports a system of stationary loads which consist of its self-weight, the weight of any permanent fixtures (such as partitions, machinery, etc.) and also a system of imposed or 'live' loads which comprise snow loads, wind loads or any movable equipment. The structural elements of the building must then be designed to withstand the worst combination of these fixed and movable loads.

Other forms of movable load consist of vehicles and trains that cross bridges and viaducts. Again, these structures must be designed to support their self-weight, the weight of any permanent fixtures such as a road deck or railway track and also the forces produced by the passage of vehicles or trains. It is then necessary to determine the critical positions of the vehicles or trains in relation to the bridge or viaduct. Although these loads are moving loads, they are assumed to be moving or changing at such a slow rate that dynamic effects (such as vibrations and oscillating stresses) are absent.

The effects of loads that occupy different positions on a structure can be studied by means of *influence lines.* Influence lines give the value at a *particular* point in a structure of functions such as shear force, bending moment and displacement for *all* positions of a travelling unit load; they may also be constructed to show the variation of support reaction with the unit load position. From these influence lines the value of a function at a point can be calculated for a system of loads traversing the structure. For this we use the principle of superposition so that the structural systems we consider must be linearly elastic.

20.1 Influence lines for beams in contact with the load

We shall now investigate the construction of influence lines for support reactions and for the shear force and bending moment at a section of a beam when the travelling load is in continuous contact with the beam.

Consider the simply supported beam AB shown in Fig. 20.1(a) and suppose that we wish to construct the influence lines for the support reactions, R_A and R_B, and also for the shear force, S_K, and bending moment, M_K, at a given section K; all the influence lines are constructed by considering the passage of a unit load across the beam.

R_A influence line

Suppose that the unit load has reached a position C, a distance x from A, as it travels across the beam. Then, considering the moment equilibrium of the beam about B we have

$$R_A L - 1(L - x) = 0$$

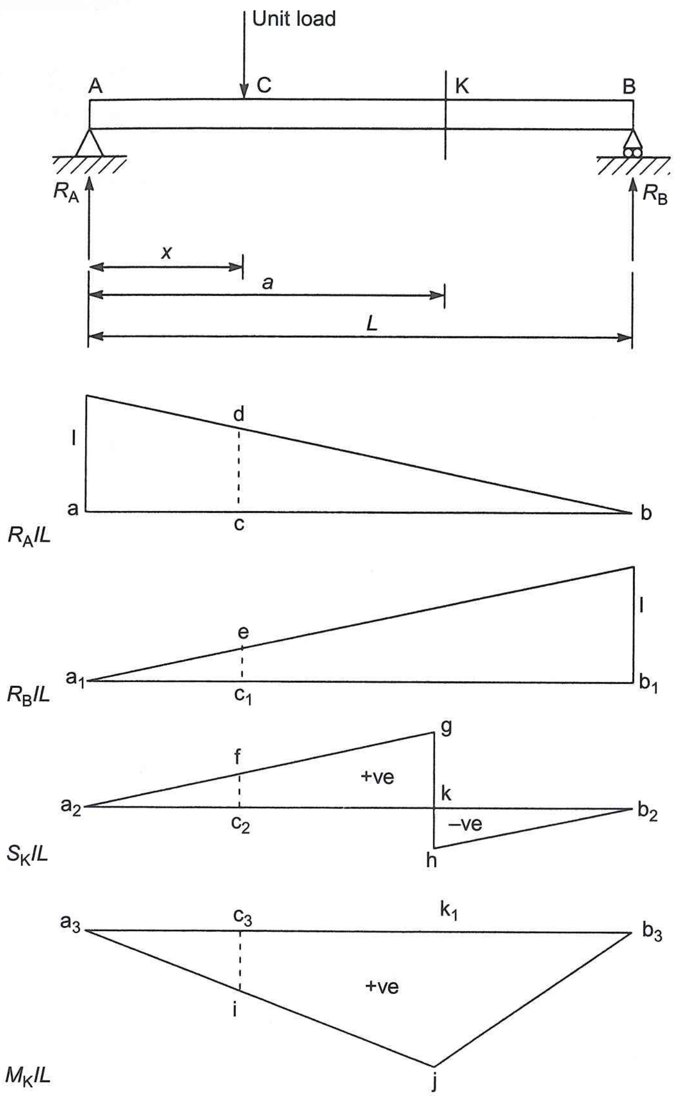


FIGURE 20.1

Reaction, shear force and bending moment influence lines for a simply supported beam.

ich gives

$$R_A = \frac{L-x}{L} \tag{20.1}$$

Hence R_A is a linear function of x and when $x=0$, $R_A=1$ and when $x=L$, $R_A=0$; both these results obvious from inspection. The influence line *(IL)* for R_A *($R_A IL$)* is then as shown in Fig. 20.1(b). Note t when the unit load is at C, the value of R_A is given by the ordinate cd in the R_A influence line.

influence line

e influence line for the reaction R_B is constructed in an identical manner. Thus, taking moments

$$R_B L - 1x = 0$$

so that

$$R_B = \frac{x}{L} \tag{20.2}$$

Equation (20.2) shows that R_B is a linear function of x. Further, when $x=0$, $R_B=0$ and when $x=L$, $R_B=1$, giving the influence line shown in Fig. 20.1(c). Again, with the unit load at C the value of R_B is equal to the ordinate c_1e in Fig. 20.1(c).

S_K influence line

The value of the shear force at the section K depends upon the position of the unit load, i.e. whether it is between A and K or between K and B. Suppose initially that the unit load is at the point C between A and K. Then the shear force at K is given by

$$S_K = R_B$$

so that from Eq. (20.2)

$$S_K = \frac{x}{L} \quad (0 \le x \le a) \tag{20.3}$$

The sign convention for shear force is that adopted in Section 3.2. We could have established Eq. (20.3) by expressing S_K in terms of R_A. Thus

$$S_K = -R_A + 1$$

Substituting for R_A from Eq. (20.1) we obtain

$$S_K = -\frac{L-x}{L} + 1 = \frac{x}{L}$$

as before. Clearly, however, expressing S_K in the terms of R_B is the most direct approach.

We see from Eq. (20.3) that S_K varies linearly with the position of the load. Therefore, when $x=0$, $S_K=0$ and when $x=a$, $S_K=a/L$, the ordinate kg in Fig. 20.1(d), and is the value of S_K with the unit load immediately to the left of K. Thus, with the load between A and K the S_K influence line is the line a_2g in Fig. 20.1(d) so that, when the unit load is at C, the value of S_K is equal to the ordinate c_2f.

With the unit load between K and B the shear force at K is given by

$$S_K = -R_A \quad (\text{or } S_K = R_B - 1)$$

Substituting for R_A from Eq. (20.1) we have

$$S_K = -\frac{L-x}{L} \quad (a \le x \le L) \tag{20.4}$$

Again S_K is a linear function of load position. Therefore when $x=L$, $S_K=0$ and when $x=a$, i.e. the

From Fig. 20.1(d) we see that the gradient of the line a_2g is equal to $[(a/L) - 0]/a = 1/L$ and that the dient of the line hb_2 is equal to $[0 + (L - a)/L]/(L - a) = 1/L$. Thus the gradient of the S_K influence : is the same on both sides of K. Furthermore, $gh = kh + kg$ or $gh = (L - a)/L + a/L = 1$.

; influence line

e value of the bending moment at K also depends upon whether the unit load is to the left or right ζ. With the unit load at C

$$M_K = R_B(L - a) \quad (\text{or } M_K = R_A\,a - 1(a - x))$$

ch, when substituting for R_B from Eq. (20.2) becomes

$$M_K = \frac{(L-a)}{L}x \quad (0 \le x \le a) \tag{20.5}$$

From Eq. (20.5) we see that M_K varies linearly with x. Therefore, when $x = 0$, $M_K = 0$ and when a, $M_K = (L - a)a/L$, which is the ordinate k_1j in Fig. 20.1(e).

Now with the unit load between K and B

$$M_K = R_A a$$

ch becomes, from Eq. (20.1)

$$M_K = \left(\frac{L-x}{L}\right)a \quad (a \le x \le L) \tag{20.6}$$

Again M_K is a linear function of x so that when $x = a$, $M_K = (L - a)a/L$, the ordinate k_1j in 20.1(e), and when $x = L$, $M_K = 0$. The complete influence line for the bending moment at K is ı the line a_3jb_3 as shown in Fig. 20.1(e). Hence the bending moment at K with the unit load at C ıe ordinate c_3i in Fig. 20.1(e).

In establishing the shear force and bending moment influence lines for the section K of the beam in 20.1(a) we have made use of the previously derived relationships for the support reactions, R_A and R_B. ıly the influence lines for S_K and M_K had been required, the procedure would have been as follows.

With the unit load between A and K

$$S_K = R_B$$

Now, taking moments about A

$$R_B L - 1x = 0$$

hat

$$R_B = \frac{x}{L}$$

Therefore

$$S_K = \frac{x}{L}$$

This, of course, amounts to the same procedure as before except that the calculation of R_B follows the writing down of the expression for S_K. The remaining equations for the influence lines for S_K and M_K are derived in a similar manner.

We note from Fig. 20.1 that all the influence lines are composed of straight-line segments. This is always the case for statically determinate structures. We shall therefore make use of this property when considering other beam arrangements.

EXAMPLE 20.1

Draw influence lines for the shear force and bending moment at the section C of the beam shown in Fig. 20.2(a).

In this example we are not required to obtain the influence lines for the support reactions. However, the influence line for the reaction R_A has been included to illustrate the difference between this influence line and the influence line for R_A in Fig. 20.1(b); the reader should verify the R_A influence line in Fig. 20.2(b).

Since we have established that influence lines for statically determinate structures consist of linear segments they may be constructed by placing the unit load at different positions, which will enable us to calculate the principal values.

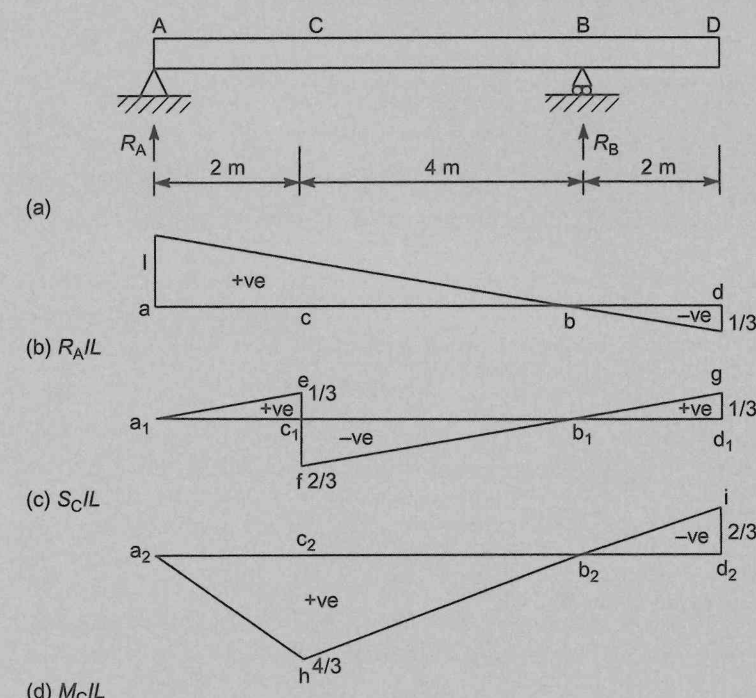


FIGURE 20.2

Shear force and bending moment influence lines for the beam of Ex. 20.1.

nfluence line

h the unit load at A

$$S_C = -R_B = 0 \quad \text{(by inspection)}$$

Vith the unit load immediately to the left of C

$$S_C = R_B \tag{i}$$

Now taking moments about A we have

$$R_B \times 6 - 1 \times 2 = 0$$

ch gives

$$R_B = \frac{1}{3}$$

Therefore, from Eq. (i)

$$S_C = \frac{1}{3} \tag{ii}$$

Now with the unit load immediately to the right of C

$$S_C = -R_A \tag{iii}$$

Taking moments about B gives

$$R_A \times 6 - 1 \times 4 = 0$$

nce

$$R_A = \frac{2}{3}$$

hat, from Eq. (iii)

$$S_C = -\frac{2}{3}$$

Vith the unit load at B

$$S_C = -R_A = 0 \quad \text{(by inspection)} \tag{iv}$$

lacing the unit load at D we have

$$S_C = -R_A \tag{v}$$

gain taking moments about B

$$R_A \times 6 + 1 \times 2 = 0$$

which

$$R_A = -\frac{1}{3}$$

Hence

$$S_C = \frac{1}{3} \tag{vi}$$

The complete influence line for the shear force at C is then as shown in Fig. 20.2(c). Note that the gradient of each of the lines a_1e, fb_1 and b_1g is the same.

M_C influence line

With the unit load placed at A

$$M_C = +R_B \times 4 = 0 \quad (R_B = 0 \text{ by inspection})$$

With the unit load at C

$$M_C = +R_A \times 2 = +\frac{4}{3}$$

in which $R_A = 2/3$ with the unit load at C (see above). With the unit load at B

$$M_C = +R_A \times 2 = 0 \quad (R_A = 0 \text{ by inspection})$$

Finally, with the unit load at D

$$M_C = +R_A \times 2$$

but, again from the calculation of S_C, $R_A = -1/3$. Hence

$$M_C = -\frac{2}{3}$$

The complete influence line for the bending moment at C is shown in Fig. 20.2(d). Note that the line hb_2i is one continuous line.

20.2 Mueller-Breslau principle

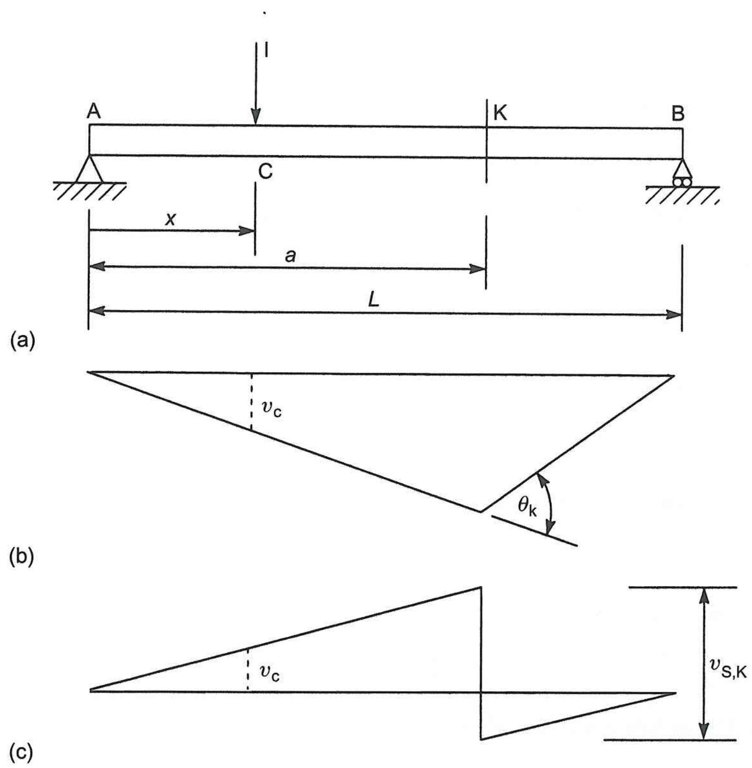
A simple and convenient method of constructing influence lines is to employ the Mueller–Breslau principle which gives the shape of an influence line without the values of its ordinates; these, however, are easily calculated for statically determinate systems from geometry.

Consider the simply supported beam, AB, shown in Fig. 20.3(a) and suppose that a unit load is crossing the beam and has reached the point C a distance x from A. Suppose also that we wish to determine the influence line for the moment at the section K, a distance a from A. We now impose a virtual displacement, v_C, at C such that internal work is done only by the moment at K, i.e. we allow a change in gradient, θ_K, at K so that the lengths AK and KB rotate as rigid links as shown in Fig. 20.3(b). Therefore, from the principle of virtual work (Chapter 15), the external virtual work done by the unit load is equal to the internal virtual work done by the moment, M_K, at K. Thus

$$1v_C = M_K\theta_K$$

If we choose v_C so that θ_K is equal to unity

$$M_K = v_C \tag{20.7}$$



20.3

ıtion of the Mueller–Breslau principle.

moment at the section K due to a unit load at the point C, an arbitrary distance x from A, is to the magnitude of the virtual displacement at C. But, as we have seen in Section 20.1, the nt at a section K due to a unit load at a point C is the influence line for the moment at K. ore, the M_K influence line may be constructed by introducing a hinge at K and imposing a unit in angle at K; the displaced shape is then the influence line.

e argument may be extended to the construction of the influence line for the shear force, S_K, at tion K. Suppose now that the virtual displacement, v_C, produces a shear displacement, $v_{S,K}$, at own in Fig. 20.3(c). Note that the direction of v_C is now in agreement with the sign convention ar force. Again, from the principle of virtual work

$$1 v_C = S_K v_{S,K}$$

we choose v_C so that $v_{S,K} = 1$

$$S_K = v_C \tag{20.8}$$

nce, since the shear force at the section K due to a unit load at any point C is the influence line shear force at K, we see that the displaced shape in Fig. 20.3(c) is the influence line for S_K the displacement at K produced by the virtual displacement at C is unity. A similar argument

The Mueller–Breslau principle demonstrated above may be stated in general terms as follows:

The shape of an influence line for a particular function (support reaction, shear force, bending moment, etc.) can be obtained by removing the resistance of the structure to that function at the section for which the influence line is required and applying an internal force corresponding to that function so that a unit displacement is produced at the section. The resulting displaced shape of the structure then represents the shape of the influence line.

EXAMPLE 20.2

Use the Mueller–Breslau principle to determine the shape of the shear force and bending moment influence lines for the section C in the beam in Ex. 20.1 (Fig. 20.2(a)) and calculate the values of the principal ordinates.

In Fig. 20.4(b) we impose a unit shear displacement at the section C. In effect we are removing the resistance to shear of the beam at C by cutting the beam at C. We then apply positive shear forces to the two faces of the cut section in accordance with the sign convention of Section 3.2. Thus the beam to the right of C is displaced downwards while the beam to the left of C is displaced upwards. Since the slope of the influence line is the same on each side of C we can determine the ordinates of the influence line by geometry. Hence, in Fig. 20.4(b)

$$\frac{c_1 e}{c_1 a_1} = \frac{c_1 f}{c_1 b_1}$$

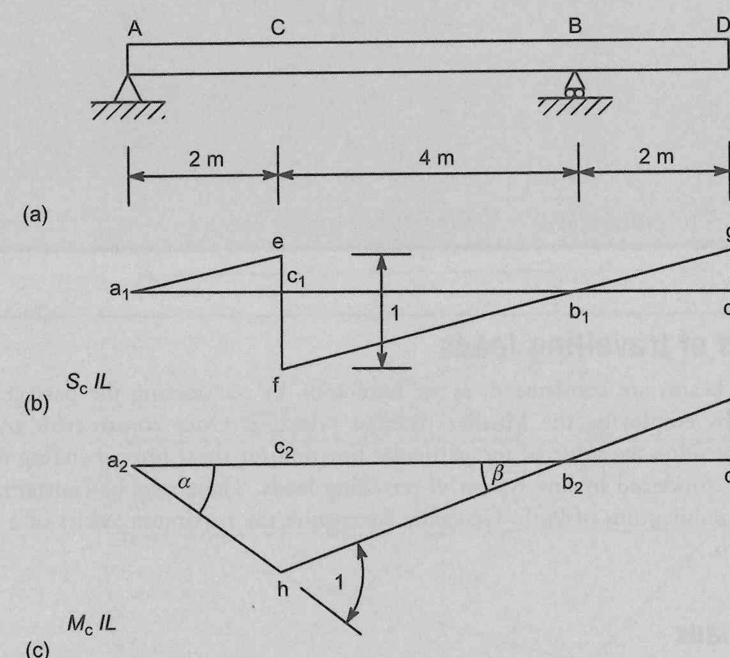


FIGURE 20.4

Construction of influence lines using the Mueller–Breslau principle.

Therefore

$$c_1e = \frac{c_1a_1}{c_1b_1}c_1f = \frac{1}{2}c_1f$$

Further, since

$$c_1e + c_1f = 1$$

$$c_1e = \frac{1}{3} \quad c_1f = \frac{2}{3}$$

; before. The ordinate d_1g $(=\frac{1}{3})$ follows.

In Fig. 20.4(c) we have, from the geometry of a triangle,

$$\alpha + \beta = 1 \text{ (external angle = sum of opposite internal angles)}$$

Then, assuming that the angles α and β are small so that their tangents are equal to the angles in ıdians

$$\frac{c_2h}{c_2a_2} + \frac{c_2h}{c_2b_2} = 1$$

r

$$c_2h\left(\frac{1}{2} + \frac{1}{4}\right) = 1$$

hence

$$c_2h = \frac{4}{3}$$

in Fig. 20.2(d). The ordinate $d_2i(=\frac{2}{3})$ follows from similar triangles.

).3 Systems of travelling loads

.uence lines for beams are constructed, as we have seen, by considering the passage of a unit load ɔss a beam or by employing the Mueller–Breslau principle. Once constructed, an influence line y be used to determine the value of the particular function for shear force, bending moment, etc. at :ction of a beam produced by any system of travelling loads. These may be concentrated loads, dis- 'uted loads or combinations of both. Generally we require the maximum values of a function as the ds cross the beam.

ncentrated loads

definition the ordinate of an influence line at a point gives the value of the function at a specified sec- ı of a beam due to a unit load positioned at the point. Thus, in the beam shown in Fig. 20.1(a) the ar force at K due to a unit load at C is equal to the ordinate c_2f in Fig. 20.1(d). Since we are assuming : the system is linear it follows that the shear force at K produced by a load W at C is $W c_2 f$.

The argument may be extended to any number of travelling loads whose positions are fixed in relation to each other. In Fig. 20.5(a), for example, three concentrated loads, W_1, W_2 and W_3 are crossing the beam AB and are at fixed distances c and d apart. Suppose that they have reached the positions C, D and E, respectively. Let us also suppose that we require values of shear force and bending moment at the section K; the S_K and M_K influence lines are then constructed using either of the methods described in Sections 20.1 and 20.2.

Since the system is linear we can use the principle of superposition to determine the combined effects of the loads. Therefore, with the loads in the positions shown, and referring to Fig. 20.5(b)

$$S_K = W_1s_1 + W_2s_2 + W_3s_3 \tag{20.9}$$

in which s_1, s_2 and s_3 are the ordinates under the loads in the S_K influence line.

Similarly, from Fig. 20.5(c)

$$M_K = W_1m_1 + W_2m_2 + W_3m_3 \tag{20.10}$$

where m_1, m_2 and m_3 are the ordinates under the loads in the M_K influence line.

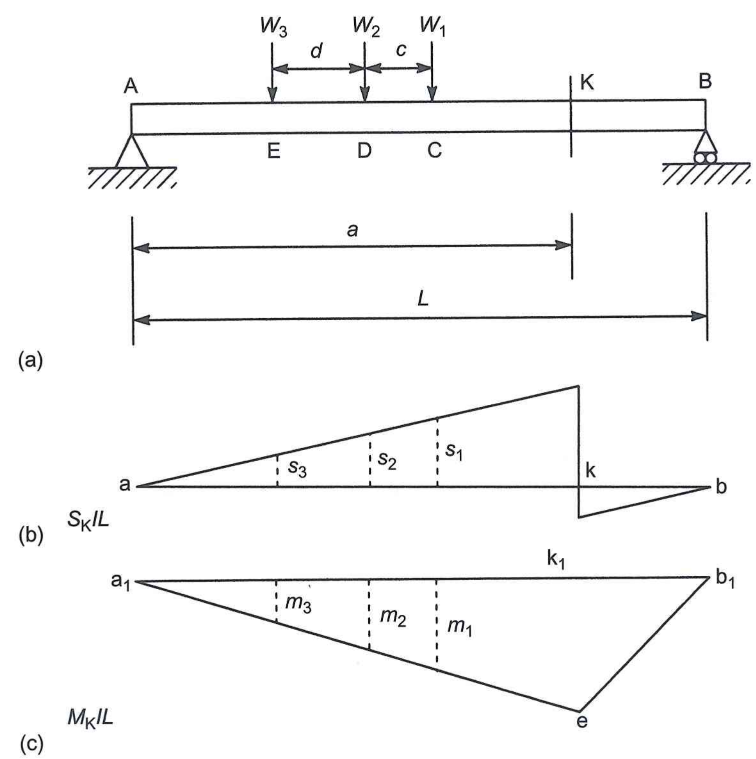


FIGURE 20.5

ximum shear force at K

ın be seen from Fig. 20.5(b) that, as the loads W_1, W_2 and W_3 move to the right, the ordinates s_1, s_2 s_3 increase in magnitude so that the shear force at K increases positively to a peak value with W_1 just ıe left of K. When W_1 passes to the right of K, the ordinate, s_1, becomes negative, then

$$S_K = -W_1 s_1 + W_2 s_2 + W_3 s_3$$

the magnitude of S_K suddenly drops. As the loads move further to the right the now negative ordi- s_1 decreases in magnitude while the ordinates s_2 and s_3 increase positively. Therefore, a second value of S_K occurs with W_2 just to the left of K. When W_2 passes to the right of K the ordinate s_2 ımes negative and

$$S_K = -W_1 s_1 - W_2 s_2 + W_3 s_3$$

hat again there is a sudden fall in the positive value of S_K. A third peak value is reached with W_3 to the left of K and then, as W_3 passes to the right of K, S_K becomes completely negative. The e arguments apply for negative values of S_K as the loads travel from right to left.

Thus we see that maximum positive and negative values of shear force at a section of a beam occur n one of the loads is at that section. In some cases it is obvious which load will give the greatest e, in other cases a trial and error method is used.

ximum bending moment at K

milar situation arises when determining the position of a set of loads to give the maximum bending nent at a section of a beam although, as we shall see, a more methodical approach than trial and r may be used when the critical load position is not obvious.

With the loads W_1, W_2 and W_3 positioned as shown in Fig. 20.5(a) the bending moment, M_K, at given by Eq. (20.10), i.e.

$$M_K = W_1 m_1 + W_2 m_2 + W_3 m_3$$

As the loads move to the right the ordinates m_1, m_2 and m_3 increase in magnitude until W_1 passes K m_1 begins to decrease. Thus M_K reaches a peak value with W_1 at K. Further movement of the loads to ight causes m_2 and m_3 to increase, while m_1 decreases so that a second peak value occurs with W_2 at milarly, a third peak value is reached with W_3 at K. Thus the maximum bending moment at K will r with a load at K. In some cases this critical load is obvious, or it may be found by trial and error as he maximum shear force at K. However, alternatively, the critical load may be found as follows.

uppose that the beam in Fig. 20.5(a) carries a system of concentrated loads, W_1, W_2, ..., W_j, ..., and that they are in any position on the beam. Then, from Eq. (20.10)

$$M_K = \sum_{j=1}^{n} W_j m_j \tag{20.11}$$

uppose now that the loads are given a small displacement δx. The bending moment at K then mes $M_K + \delta M_K$ and each ordinate m becomes $m + \delta m$. Therefore, from Eq. (20.11)

$$M_K + \delta M_K = \sum_{j=1}^{n} W_j (m_j + \delta m_j)$$

or

$$M_K + \delta M_K = \sum_{j=1}^{n} W_j m_j + \sum_{j=1}^{n} W_j \delta m_j$$

whence

$$\delta M_K = \sum_{j=1}^{n} W_j \delta m_j$$

Therefore, in the limit as $\delta x \to 0$

$$\frac{dM_K}{dx} = \sum_{j=1}^{n} W_j \frac{dm_j}{dx}$$

in which dm_j/dx is the gradient of the M_K influence line. Therefore, if

$$\sum_{j=1}^{n} W_{j,L}$$

is the sum of the loads to the left of K and

$$\sum_{j=1}^{n} W_{j,R}$$

is the sum of the loads to the right of K, we have, from Eqs (20.5) and (20.6)

$$\frac{dM_K}{dx} = \sum_{j=1}^{n} W_{j,L}\left(\frac{L-a}{L}\right) + \sum_{j=1}^{n} W_{j,R}\left(-\frac{a}{L}\right)$$

For a maximum value of M_K, $dM_K/dx = 0$ so that

$$\sum_{j=1}^{n} W_{j,L}\left(\frac{L-a}{L}\right) = \sum_{j=1}^{n} W_{j,R}\frac{a}{L}$$

or

$$\frac{1}{a}\sum_{j=1}^{n} W_{j,L} = \frac{1}{L-a}\sum_{j=1}^{n} W_{j,R} \tag{20.12}$$

From Eq. (20.12) we see that the bending moment at K will be a maximum with one of the loads at K (from the previous argument) and when the load per unit length of beam to the left of K is equal to the load per unit length of beam to the right of K. Part of the load at K may be allocated to AK and part to KB as required to fulfil this condition.

Equation (20.12) may be extended as follows. Since

$$\sum_{j=1}^{n} W_j = \sum_{j=1}^{n} W_{j,L} + \sum_{j=1}^{n} W_{j,R}$$

then

$$\sum_{j=1}^{n} W_{j,R} = \sum_{j=1}^{n} W_j - \sum_{j=1}^{n} W_{j,L}$$

bstituting for

$$\sum_{j=1}^{n} W_{j,\mathrm{R}}$$

. (20.12) we obtain

$$\frac{1}{a}\sum_{j=1}^{n} W_{j,\mathrm{L}} = \left(\frac{1}{L-a}\right)\left(\sum_{j=1}^{n} W_j - \sum_{j=1}^{n} W_{j,\mathrm{L}}\right)$$

arranging we have

$$\frac{L-a}{a} = \frac{\sum_{j=1}^{n} W_j - \sum_{j=1}^{n} W_{j,\mathrm{L}}}{\sum_{j=1}^{n} W_{j,\mathrm{L}}}$$

ce

$$\frac{1}{L}\sum_{j=1}^{n} W_j = \frac{1}{a}\sum_{j=1}^{n} W_{j,\mathrm{L}} \tag{20.13}$$

ombining Eqs (20.12) and (20.13) we have

$$\frac{1}{L}\sum_{j=1}^{n} W_j = \frac{1}{a}\sum_{j=1}^{n} W_{j,\mathrm{L}} = \frac{1}{L-a}\sum_{j=1}^{n} W_{j,\mathrm{R}} \tag{20.14}$$

herefore, for M_{K} to be a maximum, there must be a load at K such that the load per unit length he complete span is equal to the load per unit length of beam to the left of K and the load per ngth of beam to the right of K.

MPLE 20.3

rmine the maximum positive and negative values of shear force and the maximum value of ing moment at the section K in the simply supported beam AB shown in Fig. 20.6(a) when it ssed by the system of loads shown in Fig. 20.6(b).

he influence lines for the shear force and bending moment at K are constructed using either of nethods described in Sections 20.1 and 20.2 as shown in Fig. 20.6(c) and (d).

imum positive shear force at K

clear from inspection that S_{K} will be a maximum with the 5 kN load just to the left of K, in h case the 3 kN load is off the beam and the ordinate under the 4 kN load in the S_{K} influence s, from similar triangles, 0.1. Then

$$S_{\mathrm{K}}(\max) = 5 \times 0.3 + 4 \times 0.1 = 1.9\ \mathrm{kN}$$

imum negative shear force at K

e are two possible load positions which could give the maximum negative value of shear force at either can be eliminated by inspection. First we shall place the 3 kN load just to the right of K. ordinates under the 4 and 5 kN loads are calculated from similar triangles and are -0.5 and 3, respectively. Then

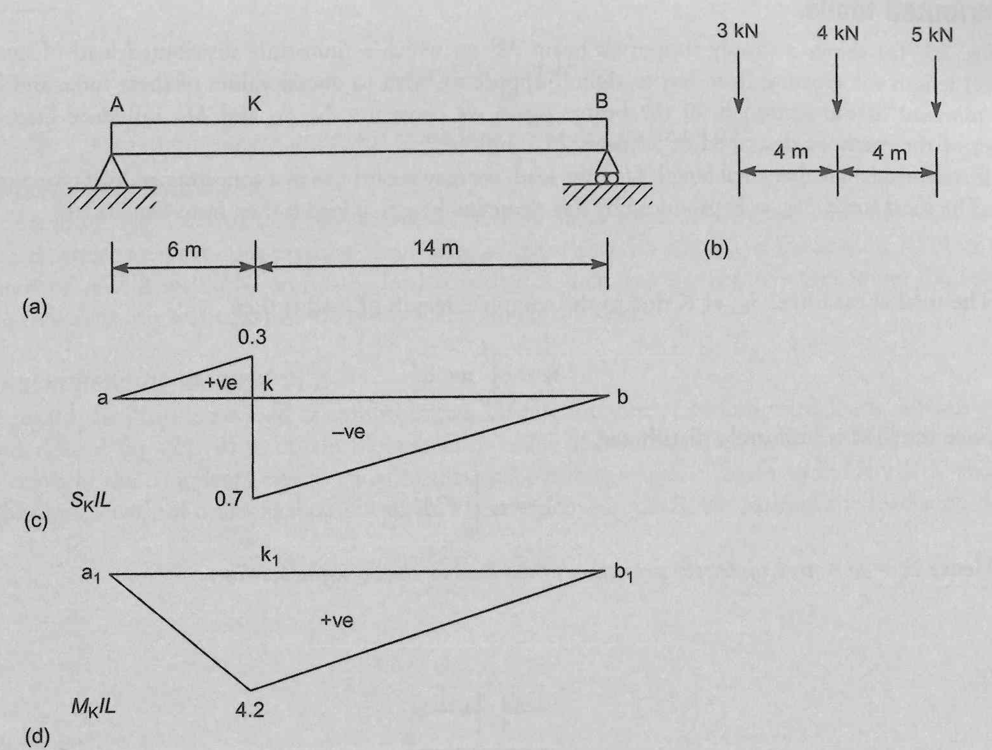


FIGURE 20.6

Determination of the maximum shear force and bending moment at a section of a beam.

$$S_{\mathrm{K}} = 3 \times (-0.7) + 4 \times (-0.5) + 5 \times (-0.3) = -5.6\ \mathrm{kN}$$

Now with the 4 kN load just to the right of K, the ordinates under the 3 and 5 kN loads are 0.1 and -0.5, respectively. Then

$$S_{\mathrm{K}} = 3 \times (0.1) + 4 \times (-0.7) + 5 \times (-0.5) = -5.0\ \mathrm{kN}$$

Therefore the maximum negative value of S_{K} is -5.6 kN and occurs with the 3 kN load immediately to the right of K.

Maximum bending moment at K

We position the loads in accordance with the criterion of Eq. (20.14). The load per unit length of the complete beam is $(3 + 4 + 5)/20 = 0.6$ kN/m. Therefore if we position the 4 kN load at K and allocate 0.6 kN of the load to AK the load per unit length on AK is $(3 + 0.6)/6 = 0.6$ kN/m and the load per unit length on KB is $(3.4 + 5)/14 = 0.6$ kN/m. The maximum bending moment at K therefore occurs with the 4 kN load at K; in this example the critical load position could have been deduced by inspection.

With the loads in this position the ordinates under the 3 and 5 kN loads in the M_{K} influence line are 1.4 and 3.0, respectively. Then

$$M_{\mathrm{K}}(\max) = 3 \times 1.4 + 4 \times 4.2 + 5 \times 3.0 = 36.0\ \mathrm{kNm}$$

stributed loads

ure 20.7(a) shows a simply supported beam AB on which a uniformly distributed load of intensity ınd length l is crossing from left to right. Suppose we wish to obtain values of shear force and bend- moment at the section K of the beam. Again we construct the S_K and M_K influence lines using ıer of the methods described in Sections 20.1 and 20.2.

If we consider an elemental length δl of the load, we may regard this as a concentrated load of magnitude . The shear force, δS_k, at K produced by this elemental length of load is then from Fig. 20.7(b)

$$\delta S_K = w\delta l s$$

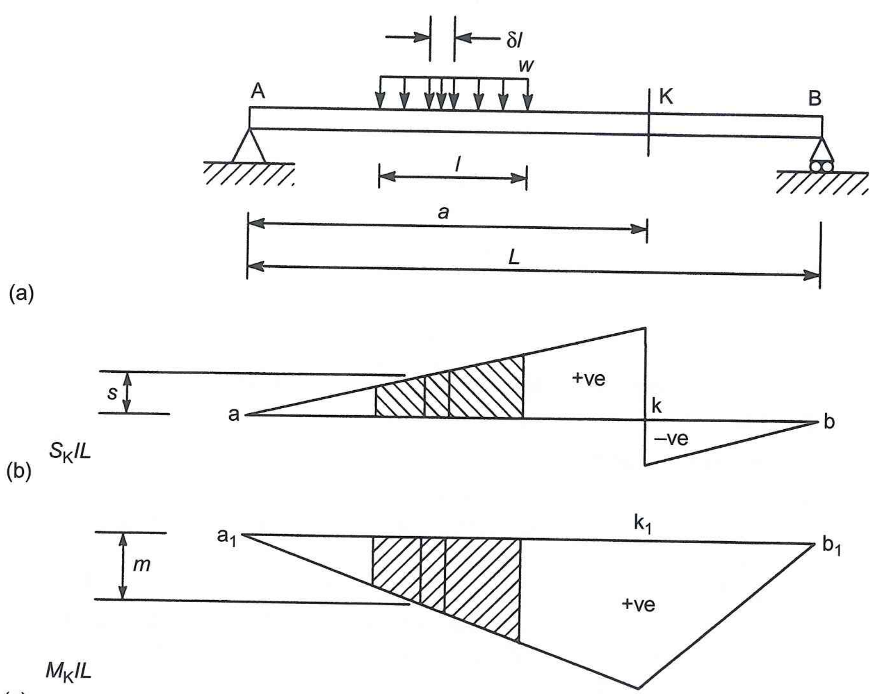
The total shear force, S_K, at K due to the complete length of load is then

$$S_K = \int_0^l ws \, dl$$

since the load is uniformly distributed

$$S_K = w\int_0^l s \, dl \tag{20.15}$$

Hence $S_k = w \times$ *area under the projection of the load in the* S_K *influence line.*



E 20.7

force and bending moment due to a ...

Similarly

$$M_K = w\int_0^l m \, dl \tag{20.16}$$

so that $M_k = w \times$ *area under the projection of the load in the* M_K *influence line.*

Maximum shear force at K

It is clear from Fig. 20.7(b) that the maximum positive shear force at K occurs with the head of the load at K while the maximum negative shear force at K occurs with the tail of the load at K. Note that the shear force at K would be zero if the load straddled K such that the negative area under the load in the S_K influence line was equal to the positive area under the load.

Maximum bending moment at K

If we regard the distributed load as comprising an infinite number of concentrated loads, we can apply the criterion of Eq. (20.14) to obtain the maximum value of bending moment at K. Thus the load per unit length of the complete beam is equal to the load per unit length of beam to the left of K and the load per unit length of beam to the right of K. Therefore, in Fig. 20.8, we position the load such that

$$\frac{w\text{ck}_1}{\text{a}_1\text{k}_1} = \frac{w\text{dk}_1}{\text{k}_1\text{b}_1}$$

or

$$\frac{\text{ck}_1}{\text{a}_1\text{k}_1} = \frac{\text{dk}_1}{\text{k}_1\text{b}_1} \tag{20.17}$$

From Fig. 20.8

$$\frac{\text{fc}}{\text{hk}_1} = \frac{\text{a}_1\text{c}}{\text{a}_1\text{k}_1}$$

so that

$$\text{fc} = \frac{\text{a}_1\text{c}}{\text{a}_1\text{k}_1}\text{hk}_1 = \left(\frac{\text{a}_1\text{k}_1 - \text{ck}_1}{\text{a}_1\text{k}_1}\right)\text{hk}_1 = \left(1 - \frac{\text{ck}_1}{\text{a}_1\text{k}_1}\right)\text{hk}_1$$

Similarly

$$\text{dg} = \left(1 - \frac{\text{dk}_1}{\text{b}_1\text{k}_1}\right)\text{hk}_1$$

Therefore, from Eq. (20.17) we see that

$$\text{fc} = \text{dg}$$

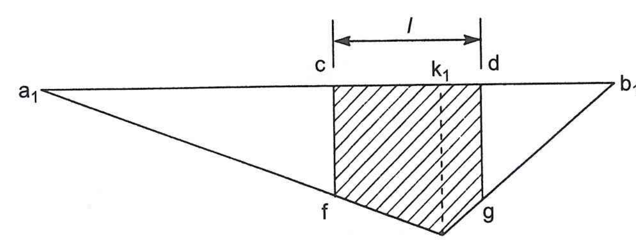


FIGURE 20.8

Load position for maximum bending

ıe ordinates under the extremities of the load in the M_K influence line are equal. It may also be ı that the area under the load in the M_K influence line is a maximum when fc = dg. This is an ıtive method of deducing the position of the load for maximum bending moment at K. Note rom Eq. (20.17), K divides the load in the same ratio as it divides the span.

MPLE 20.4

ad of length 2 m and intensity 2 kN/m crosses the simply supported beam AB shown in 20.9(a). Calculate the maximum positive and negative values of shear force and the maximum of bending moment at the quarter span point.

he shear force and bending moment influence lines for the quarter span point K are constructed e same way as before and are shown in Fig. 20.9(b) and (c).

imum shear force at K

maximum positive shear force at K occurs with the head of the load at K. In this position the ıate under the tail of the load is 0.05. Hence

$$S_K(\text{max}+\text{ve}) = 2 \times \frac{1}{2}(0.05 + 0.25) \times 2 = 0.6 \text{ kN}$$

ıe maximum negative shear force at K occurs with the tail of the load at K. With the load in ›osition the ordinate under the head of the load is −0.55. Thus

$$S_K(\text{max}-\text{ve}) = -2 \times \frac{1}{2}(0.75 + 0.55) \times 2 = -2.6 \text{ kN}$$

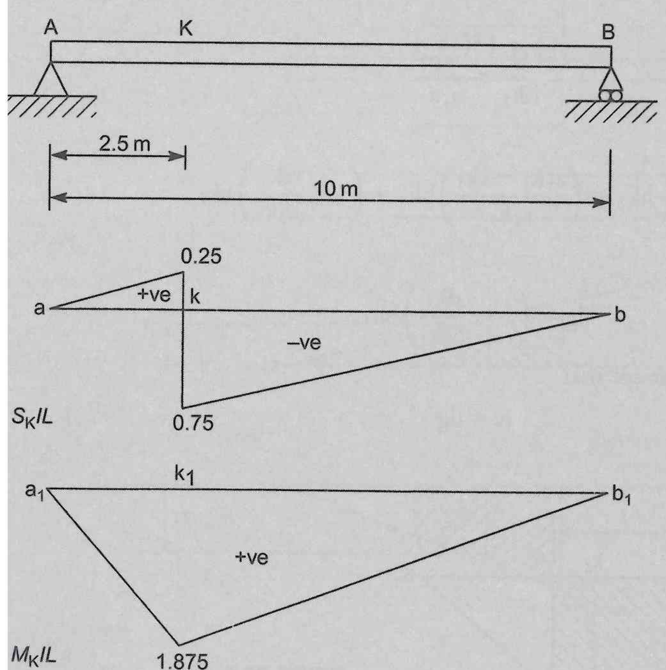


FIGURE 20.9
Maximum shear force and bending moment at the quarter span point in the beam of Ex. 20.4.

Maximum bending moment at K

We position the load so that K divides the load in the same ratio that it divides the span. Therefore 0.5 m of the load is to the left of K and 1.5 m to the right of K. The ordinate in the M_K influence line under the tail of the load is then 1.5 as is the ordinate under the head of the load. The maximum value of M_K is thus given by

$$M_K(\text{max}) = 2\left[\frac{1}{2}(1.5 + 1.875) \times 0.5 + \frac{1}{2}(1.875 + 1.5) \times 1.5\right]$$

which gives

$$M_K(\text{max}) = 6.75 \text{ kN m}$$

Diagram of maximum shear force

Consider the simply supported beam shown in Fig. 20.10(a) and suppose that a uniformly distributed load of intensity w and length $L/5$ (any fraction of L may be chosen) is crossing the beam. We can draw a series of influence lines for the sections, A, K_1, K_2, K_3, K_4 and B as shown in Fig. 20.10(b) and then determine the maximum positive and negative values of shear force at each of the sections K_1, K_2, etc. by considering first the head of the load at K_1, K_2, etc. and then the tail of the load at A, K_1, K_2, etc. These values are then plotted as shown in Fig. 20.10(c).

With the head of the load at K_1, K_2, K_3, K_4 and B the maximum positive shear force is given by $w(ak_1)s_1$, $w(k_1k_2)s_2$, and so on, where s_1, s_2, etc. are the mid-ordinates of the areas ak_1, k_1k_2, etc. Since s_1, s_2, etc. increase linearly, the maximum positive shear force also increases linearly at all sections of the beam between K_1 and B. At a section between A and K_1, the complete length of load will not be on the beam so that the maximum value of positive shear force at this section will not lie on the straight line and the

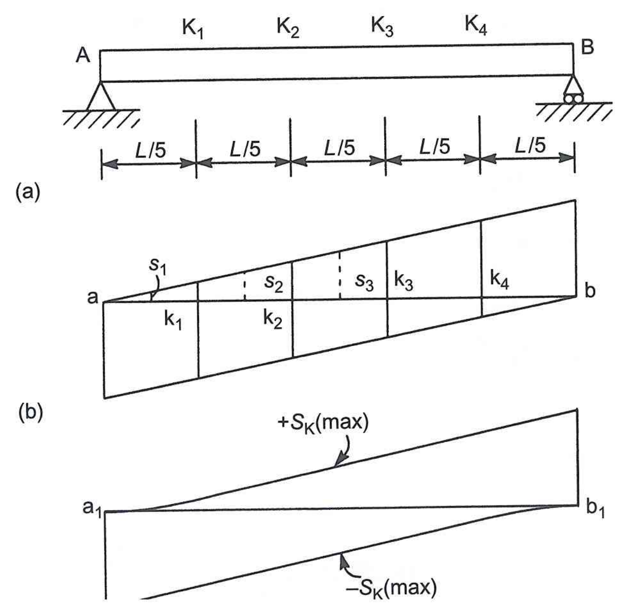


FIGURE 20.10

am of maximum positive shear force between A and K_1 will be curved; the maximum positive shear should be calculated for at least one section between A and K_1.

n identical argument applies to the calculation of the maximum negative shear force which occurs the tail of the load at a beam section. Thus, in this case, the non-linearity will occur as the load s to leave the beam between K_4 and B.

ərsal of shear force

me structures it is beneficial to know in which parts of the structure, if any, the maximum shear changes sign. In Section 4.5, for example, we saw that the diagonals of a truss resist the shear ; and therefore could be in tension or compression depending upon their orientation and the sign : shear force. If, therefore, we knew that the sign of the shear force would remain the same under esign loading in a particular part of a truss we could arrange the inclination of the diagonals so hey would always be in tension and would not be subject to instability produced by compressive . If, at the same time, we knew in which parts of the truss the shear force could change sign we introduce counterbracing (see Section 20.5).

ɔnsider the simply supported beam AB shown in Fig. 20.11(a) and suppose that it carries a uniy distributed dead load (self-weight, etc.) of intensity w_{DL}. The shear force due to this dead load lead load shear (DLS)) varies linearly from $-w_{DL}L/2$ at A to $+w_{DL}L/2$ at B as shown in !0.11(b). Suppose now that a uniformly distributed live load of length less than the span AB s the beam. As for the beam in Fig. 20.10, we can plot diagrams of maximum positive and negaıear force produced by the live load; these are also shown in Fig. 20.11(b). Then, at any section beam, the maximum shear force is equal to the sum of the maximum positive shear force due to 'e load and the DLS force, or the sum of the maximum negative shear force due to the live load ıe DLS force. The variation in this maximum shear force along the length of the beam will be easily understood if we invert the DLS force diagram.

ferring to Fig. 20.11(b) we see that the sum of the maximum negative shear force due to the live nd the DLS force is always negative between a and c. Furthermore, between a and c, the sum of aximum positive shear force due to the live load and the DLS force is always negative. Similarly,

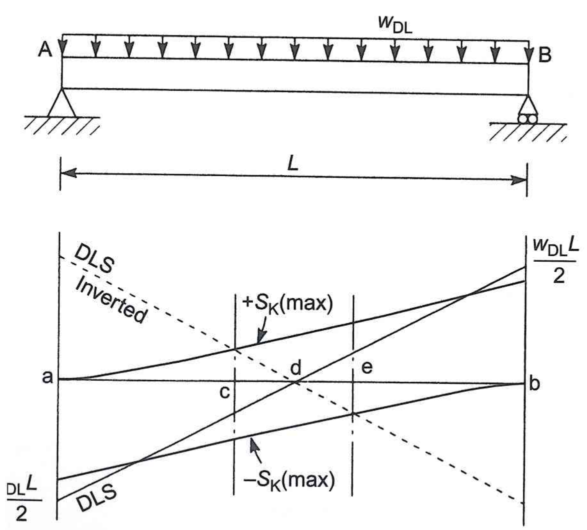


FIGURE 20.11

between e and b the maximum shear force is always positive. However, between c and e the summation of the maximum negative shear force produced by the live load and the DLS force is negative, while the summation of the maximum positive shear force due to the live load and the DLS force is positive. Therefore the maximum shear force between c and e may be positive or negative, i.e. there is a possible *reversal* of maximum shear force in this length of the beam.

EXAMPLE 20.5

A simply supported beam AB has a span of 5 m and carries a uniformly distributed dead load of 0.6 kN/m (Fig. 20.12(a)). A similarly distributed live load of length greater than 5 m and intensity 1.5 kN/m travels across the beam. Calculate the length of beam over which reversal of shear force occurs and sketch the diagram of maximum shear force for the beam.

The shear force at a section of the beam will be a maximum with the head or tail of the load at that section. Initially, before writing down an expression for shear force, we require the support reaction at A, R_A. Thus, with the head of the load at a section a distance x from A, the reaction, R_A, is found by taking moments about B.

Thus

$$R_A \times 5 - 0.6 \times 5 \times 2.5 - 1.5x\left(5 - \frac{x}{2}\right) = 0$$

whence

$$R_A = 1.5 + 1.5x - 0.15x^2 \qquad \text{(i)}$$

The maximum shear force at the section is then

$$S(\text{max}) = -R_A + 0.6x + 1.5x \qquad \text{(ii)}$$

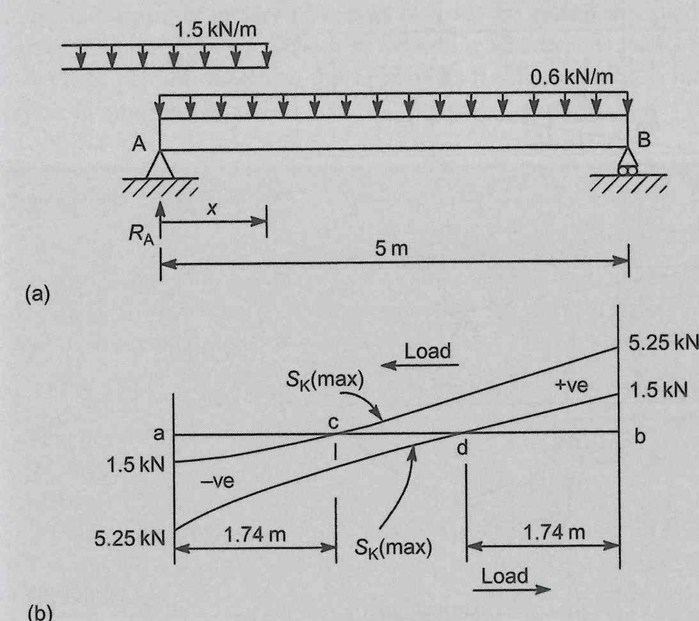


FIGURE 20.12
Reversal of shear force in the beam of Ex. 20.5.

r, substituting in Eq. (ii) for R_A from Eq. (i)

$$S(\max) = -1.5 + 0.6x + 0.15x^2 \tag{iii}$$

Equation (iii) gives the maximum shear force at any section of the beam with the load moving om left to right. Then, when $x = 0$, $S(\max) = -1.5$ kN and when $x = 5$ m, $S(\max) = +5.25$ kN. urthermore, from Eq. (iii) $S(\max) = 0$ when $x = 1.74$ m.

The maximum shear force for the load travelling from right to left is found in a similar manner. he final diagram of maximum shear force is shown in Fig. 20.12(b) where we see that reversal of iear force may take place within the length cd of the beam; cd is sometimes called the *focal length*.

termination of the point of maximum bending moment in a beam

viously we have been concerned with determining the position of a set of loads on a beam that would duce the maximum bending moment at a given section of the beam. We shall now determine the sec- ı and the position of the loads for the bending moment to be the absolute maximum.

Consider a section K a distance x_1 from the mid-span of the beam in Fig. 20.13 and suppose that a of loads having a total magnitude W_T is crossing the beam. The bending moment at K will be a ximum when one of the loads is at K; let this load be W_j. Also, suppose that the centre of gravity of complete set of loads is a distance c from the load W_j and that the total weight of all the loads to left of W_j is W_L, acting at a distance a from W_j; a and c are fixed values for a given set of loads. Initially we find R_A by taking moments about B. Hence

$$R_A L - W_T\left(\frac{L}{2} - x_1 + c\right) = 0$$

ch gives

$$R_A = \frac{W_T}{L}\left(\frac{L}{2} - x_1 + c\right)$$

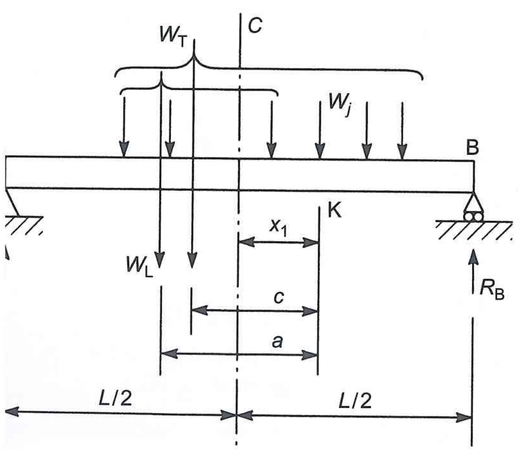


FIGURE 20.13

Determination of the absolute maximum bending

The bending moment, M_K, at K is then given by

$$M_K = R_A\left(\frac{L}{2} + x_1\right) - W_L a$$

or, substituting for R_A

$$M_A = \frac{W_T}{L}\left(\frac{L}{2} - x_1 + c\right)\left(\frac{L}{2} + x_1\right) - W_L a$$

Differentiating M_K with respect to x_1 we have

$$\frac{dM_K}{dx_1} = \frac{W_T}{L}\left[-1\left(\frac{L}{2} + x_1\right) + 1\left(\frac{L}{2} - x_1 + c\right)\right]$$

or

$$\frac{dM_K}{dx_1} = \frac{W_T}{L}(-2x_1 + c)$$

For a maximum value of M_K, $dM_K/dx_1 = 0$ so that

$$x_1 = \frac{c}{2} \tag{20.18}$$

Therefore the maximum bending moment occurs at a section K under a load W_j such that the section K and the centre of gravity of the complete set of loads are positioned at equal distances either side of the mid-span of the beam.

To apply this rule we select one of the larger central loads and position it over a section K such that K and the centre of gravity of the set of loads are placed at equal distances on either side of the mid-span of the beam. We then check to determine whether the load per unit length to the left of K is equal to the load per unit length to the right of K. If this condition is not satisfied, another load and another section K must be selected.

EXAMPLE 20.6

The set of loads shown in Fig. 20.14(b) crosses the simply supported beam AB shown in Fig. 20.14(a). Calculate the position and magnitude of the maximum bending moment in the beam.

The first step is to find the position of the centre of gravity of the set of loads. Thus, taking moments about the load W_5 we have

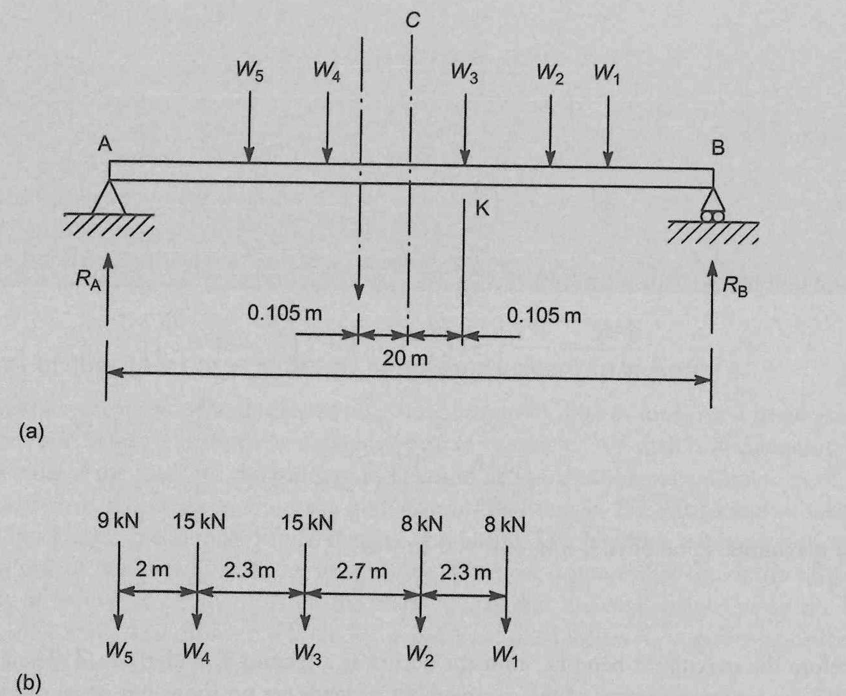
$$(9 + 15 + 15 + 8 + 8)\bar{x} = 15 \times 2 + 15 \times 4.3 + 8 \times 7.0 + 8 \times 9.3$$

whence

$$\bar{x} = 4.09 \text{ m}$$

Therefore the centre of gravity of the loads is 0.21 m to the left of the load W_3.

By inspection of Fig. 20.14(b) we see that it is probable that the maximum bending moment will occur under the load W_3. We therefore position W_3 and the centre of gravity of the set of loads at equal distances either side of the mid-span of the beam as shown in Fig. 20.14(a). We now check to determine whether this position of the loads satisfies the load per unit length condition. The load



URE 20.14

etermination of absolute maximum bending moment in the beam of Ex. 20.6.

unit length on AB = 55/20 = 2.75 kN/m. Therefore the total load required on = 2.75 × 10.105 = 27.79 kN. This is satisfied by W_5, W_4 and part (3.79 kN) of W_3.

Having found the load position, the bending moment at K is most easily found by direct calcula-. Thus taking moments about B we have

$$R_A \times 20 - 55 \times 10.105 = 0$$

ch gives

$$R_A = 27.8 \text{ kN}$$

hat

$$M_K = 27.8 \times 10.105 - 9 \times 4.3 - 15 \times 2.3 = 207.7 \text{ kN m}$$

t is possible that in some load systems there may be more than one load position which satisfies criteria for maximum bending moment but the corresponding bending moments have different es. Generally the absolute maximum bending moment will occur under one of the loads between h the centre of gravity of the system lies. If the larger of these two loads is closer to the centre ravity than the other, then this load will be the critical load; if not then both cases must be ysed.

20.4 Influence lines for beams not in contact with the load

In many practical situations, such as bridge construction for example, the moving loads are not in direct contact with the main beam or girder. Figure 20.15 shows a typical bridge construction in which the deck is supported by stringers that are mounted on cross beams which, in turn, are carried by the main beams or girders. The deck loads are therefore transmitted via the stringers and cross beams to the main beams. Generally, in the analysis, we assume that the segments of the stringers are simply supported at each of the cross beams. In Fig. 20.15 the portion of the main beam between the cross beams, for example FG, is called a *panel* and the points F and G are called *panel points*.

Figure 20.16 shows a simply supported main beam AB which supports a bridge deck via an arrangement of cross beams and stringers. Let us suppose that we wish to construct shear force and bending moment influence lines for the section K of the main beam within the panel CD. As before we consider the passage of a unit load; in this case, however, it crosses the bridge deck.

S_K influence line

With the unit load outside and to the left of the panel CD (position 1) the shear force, S_K, at K is given by

$$S_K = R_B = \frac{x_1}{L} \tag{20.19}$$

S_K therefore varies linearly as the load moves from A to C. Thus, from Eq. (20.19), when $x_1 = 0$, $S_K = 0$ and when $x_1 = a$, $S_K = a/L$, the ordinate cf in the S_K influence line shown in Fig. 20.16(b). Furthermore, from Fig. 20.16(a) we see that $S_K = S_C = S_D$ with the load between A and C, so that for a given position of the load the shear force in the panel CD has the same value at all sections.

Suppose now that the unit load is to the right of D between D and B (position 2). Then

$$S_K = -R_A = -\frac{L - x_2}{L} \tag{20.20}$$

and is linear. Therefore when $x_2 = L$, $S_K = 0$ and when $x_2 = e$, $S_K = -(L - e)/L$, the ordinate dh in the S_K influence line. Also, with the unit load between D and B, $S_K = S_C = S_D$ $(= -R_A)$ so that for a given position of the load, the shear force in the panel CD has the same value at all sections.

Now consider the unit load at some point between C and D (position 3). There will now be reaction forces, R_C and R_D, as shown in Fig. 20.16(a) acting on the stringer and the beam where, by considering the portion of the stringer immediately above the panel CD as a simply supported beam, we see that $R_C = (e - x_3)/c$ and $R_D = (x_3 - a)/c$. Therefore the shear force at K is given by

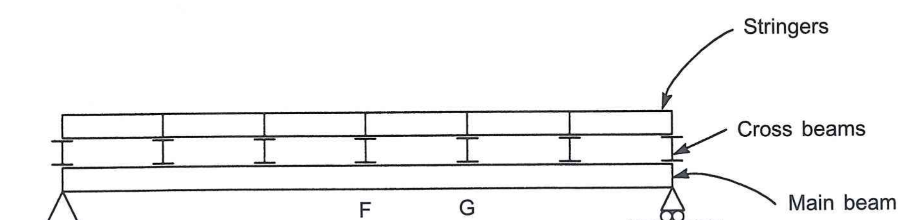
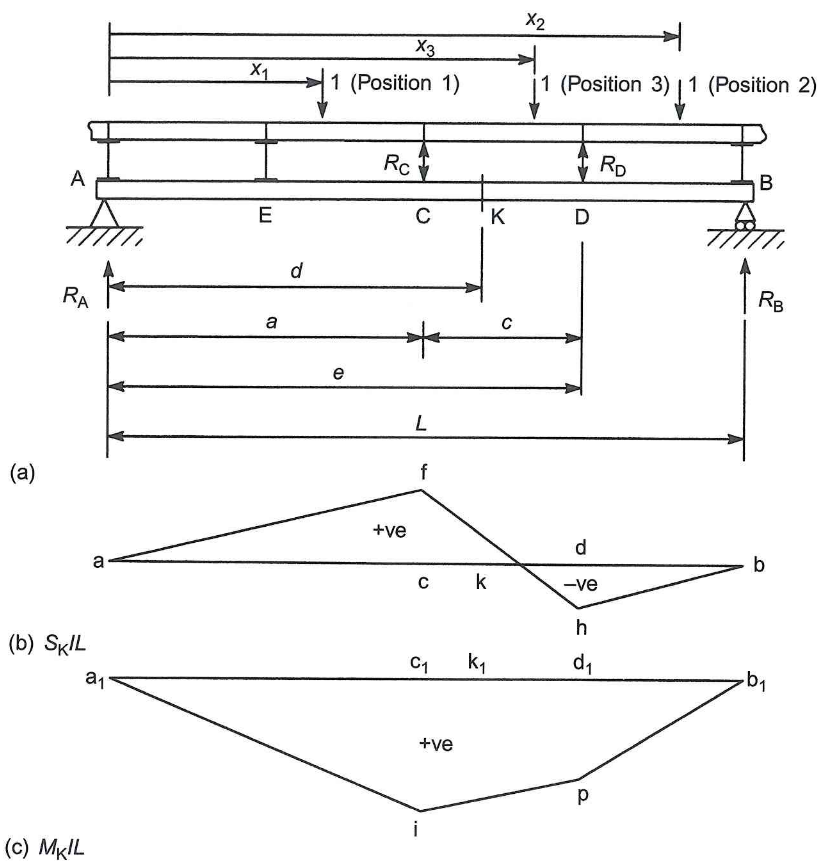


FIGURE 20.15



E 20.16

nce lines for a beam not in direct contact with the moving load.

$$S_K = R_B - R_D \quad (\text{or } S_K = -R_A + R_C)$$

ıt

$$S_K = \frac{x_3}{L} - \frac{(x_3 - a)}{c} \tag{20.21}$$

: therefore varies linearly as the load moves between C and D. Furthermore, when $x_3 = a$, $S_K = a/L$, :dinate cf in the S_K influence line, and when $x_3 = e$, $S_K = -(L - e)/L$, the ordinate dh in the S_K nce line. Note that in the calculation of the latter value, $e - a = c$.

ote also that for all positions of the unit load between C and D, $S_K = R_B + R_D$ which is indepen- ɔf the position of K. Therefore, for a given load position between C and D, the shear force is the at all sections of the panel.

ıfluence line

the unit load in position 1 between A and C, the bending moment, M_K, at K is given by

$$M_K = R_B(L - d) = \frac{x_1}{L}(L - d) \tag{20.22}$$

M_K therefore varies linearly with the load position between A and C. Also, when $x_1 = 0$, $M_K = 0$ and when $x_1 = a$, $M_K = a(L - d)/L$, the ordinate c_1i in the M_K influence line in Fig. 20.16(c).

With the unit load in position 2 between D and B

$$M_K = R_A d = \frac{L - x_2}{L} d \tag{20.23}$$

Again, M_K varies linearly with load position so that when $x_2 = e$, $M_K = (L - e)d/L$, the ordinate d_1p in the M_K influence line. Furthermore, when $x_2 = L$, $M_K = 0$.

When the unit load is between C and D (position 3)

$$M_K = R_B(L - d) - R_D(e - d)$$

As before we consider the stringer over the panel CD as a simply supported beam so that $R_D = (x_3 - a)/c$. Then since

$$R_B = \frac{x_3}{L}$$

$$M_K = \frac{x_3}{L}(L - d) - \left(\frac{x_3 - a}{c}\right)(e - d) \tag{20.24}$$

Equation (20.24) shows that M_K varies linearly with load position between C and D. Therefore, when $x_3 = a$, $M_K = a(L - d)/L$, the ordinate c_1i in the M_K influence line, and when $x_3 = e$, $M_K = d(L - e)/L$, the ordinate d_1p in the M_K influence line. Note that in the latter calculation $e - a = c$.

Maximum values of S_K and M_K

In determining maximum values of shear force and bending moment at a section of a beam that is not in direct contact with the load, certain points are worthy of note.

1. When the section K coincides with a panel point (C or D, say) the S_K and M_K influence lines are identical in geometry to those for a beam that is in direct contact with the moving load; the same rules governing maximum and minimum values therefore apply.
2. The absolute maximum value of shear force will occur in an end panel, AE or DB, when the S_K influence line will be identical in form to the bending moment influence line for a section in a simply supported beam that is in direct contact with the moving load. Therefore the same criteria for load positioning may be used for determining the maximum shear force, i.e. the load per unit length of beam is equal to the load per unit length to the left of E or D and the load per unit length to the right of E or D.
3. To obtain maximum values of shear force and bending moment in a panel, a trial and error method is the simplest approach remembering that, for concentrated loads, a load must be placed at the point where the influence line changes slope.

20.5 Forces in the members of a truss

In some instances the main beams in a bridge are trusses, in which case the cross beams are positioned at the joints of the truss. The shear force and bending moment influence lines for a panel of the truss may then be used to determine the variation in the truss member forces as moving loads cross the bridge.

Consider the simply supported Warren truss shown in Fig. 20.17(a) and suppose that it carries cross ed and the cross beams supported by the lower chord joints; the bridge deck is then the *through* type. ›se also that we wish to determine the forces in the members CD, CE, DE and GE of the truss.

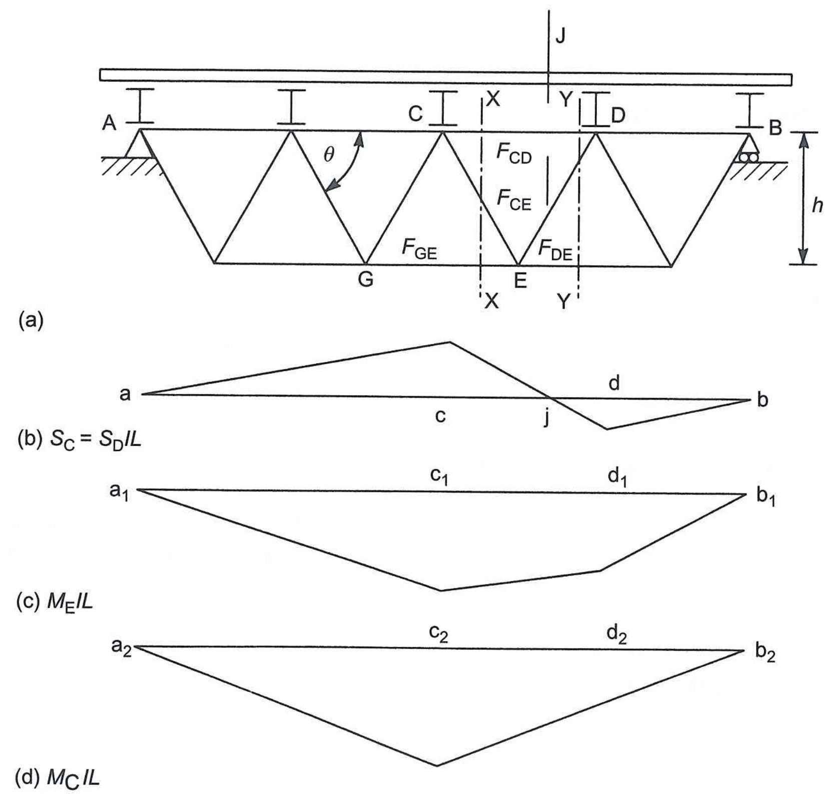
'e have seen in Section 4.5 the mechanism by which a truss resists shear forces and bending ›nts. Thus shear forces are resisted by diagonal members, while bending moments are generally d by a combination of both diagonal and horizontal members. Therefore, referring to Fig. 20.17(a), › that the forces in the members CE and DE may be determined from the shear force in the panel vhile the forces in the members CD and GE may be found from the bending moments at E and C, tively. Therefore we construct the influence lines for the shear force in the panel CD and for the ng moment at E and C, as shown in Fig. 20.17(b), (c) and (d).

Section 20.4 we saw that, for a given load position, the shear force in a panel such as CD is con- at all sections in the panel; we will call this shear force S_{CD}. Then, considering a section XX ›h CE, CD and GE, we have

$$F_{CE} \sin\theta = S_{CD}$$

t

$$F_{CE} = \frac{S_{CD}}{\sin\theta} \tag{20.25}$$



20.17

Similarly

$$F_{DE} = \frac{S_{CD}}{\sin\theta} \tag{20.26}$$

From Fig. 20.17(b) we see that for a load position between A and J, S_{CD} is positive. Therefore, referring to Fig. 20.17(a), F_{CE} is compressive while F_{DE} is tensile. For a load position between J and B, S_{CD} is negative so that F_{CE} is tensile and F_{DE} is compressive. Thus F_{CE} and F_{DE} will always be of opposite sign; this may also be deduced from a consideration of the vertical equilibrium of joint E.

If we now consider the moment equilibrium of the truss at a vertical section through joint E we have

$$F_{CD}h = M_E$$

or

$$F_{CD} = \frac{M_E}{h} \tag{20.27}$$

Since M_E is positive for all load positions (Fig. 20.17(c)), F_{CD} is compressive.

The force in the member GE is obtained from the M_C influence line in Fig. 20.17(d). Thus

$$F_{GE}h = M_C$$

which gives

$$F_{GE} = \frac{M_C}{h} \tag{20.28}$$

F_{GE} will be tensile since M_C is positive for all load positions.

It is clear from Eqs (20.25)–(20.28) that the influence lines for the forces in the members could be constructed from the appropriate shear force and bending moment influence lines. Thus, for example, the influence line for F_{CE} would be identical in shape to the shear force influence line in Fig. 20.17(b) but would have the ordinates factored by $1/\sin\theta$ and the signs reversed. The influence line for F_{DE} would also have the S_{CD} influence line ordinates factored by $1/\sin\theta$.

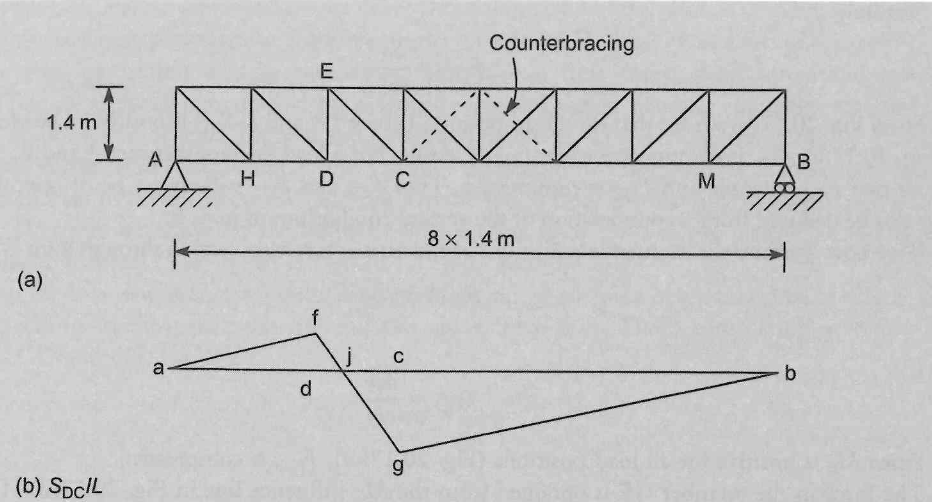
EXAMPLE 20.7

Determine the maximum tensile and compressive forces in the member EC in the Pratt truss shown in Fig. 20.18(a) when it is crossed by a uniformly distributed load of intensity 2.5 kN/m and length 4 m; the load is applied on the bottom chord of the truss.

The vertical component of the force in the member EC resists the shear force in the panel DC. Therefore we construct the shear force influence line for the panel DC as shown in Fig. 20.18(b). From Eq. (20.19) the ordinate $df = 2 \times 1.4/(8 \times 1.4) = 0.25$ while from Eq. (20.20) the ordinate $cg = (8 \times 1.4 - 3 \times 1.4)/(8 \times 1.4) = 0.625$. Furthermore, we see that S_{DC} changes sign at the point j (Fig. 20.18(b)) where jd, from similar triangles, is 0.4.

The member EC will be in compression when the shear force in the panel DC is positive and its maximum value will occur when the head of the load is at j, thereby completely covering the length aj in the S_{DC} influence line. Therefore

$$F_{EC}\sin 45^\circ = S_{DC} = 2.5 \times \frac{1}{2} \times 3.2 \times 0.25$$



IRE 20.18

ermination of the force in a member of the Pratt girder of Ex. 20.7.

which

$$F_{EC} = 1.41 \text{ kN} \quad \text{(compression)}$$

e force in the member EC will be tensile when the shear force in the panel DC is negative. fore to find the maximum tensile value of F_{EC} we must position the load within the part jb of DC influence line such that the maximum value of S_{DC} occurs. Since the positive portion of the nfluence line is triangular, we may use the criterion previously established for maximum bendoment. Thus the load per unit length over jb must be equal to the load per unit length over jc he load per unit length over cb. In other words, c divides the load in the same ratio that it s jb, i.e. 1 : 7. Therefore 0.5 m of the load is to the left of c, 3.5 m to the right. The ordinates the extremities of the load in the S_{DC} influence line are then both 0.3125 m. Hence the maxinegative shear force in the panel CD is

$$S_{CD}(\text{max} - \text{ve}) = 2.5\left[\frac{1}{2}(0.3125 + 0.625)0.5 + \frac{1}{2}(0.625 + 0.3125)3.5\right]$$

gives

$$S_{CD}(\text{max} - \text{ve}) = 4.69 \text{ kN}$$

en, since

$$F_{EC} \sin 45° = S_{CD}$$

$$F_{EC} = 6.63 \text{ kN}$$

is the maximum tensile force in the member EC.

Counterbracing

A diagonal member of a Pratt truss will, as we saw for the member EC in Ex. 20.7, be in tension or compression depending on the sign of the shear force in the particular panel in which the member is placed. The exceptions are the diagonals in the end panels where, in the Pratt truss of Fig. 20.18(a), construction of the shear force influence lines for the panels AH and MB shows that the shear force in the panel AH is always negative and that the shear force in the panel MB is always positive; the diagonals in these panels are therefore always in tension.

In some situations the diagonal members are unsuitable for compressive forces so that *counterbracing* is required. This consists of diagonals inclined in the opposite direction to the original diagonals as shown in Fig. 20.18(a) for the two centre panels. The original diagonals are then assumed to be carrying zero force while the counterbracing is in tension.

It is clear from Ex. 20.7 that the shear force in all the panels, except the two outer ones, of a Pratt truss can be positive or negative so that all the diagonals in these panels could experience compression. Therefore it would appear that all the interior panels of a Pratt truss require counterbracing. However, as we saw in Section 20.3, the dead load acting on a beam has a beneficial effect in that it reduces the length of the beam subjected to shear reversal. This, in turn, will reduce the number of panels requiring counterbracing.

EXAMPLE 20.8

The Pratt truss shown in Fig. 20.19(a) carries a dead load of 1.0 kN/m applied at its upper chord joints. A uniformly distributed live load, which exceeds 9 m in length, has an intensity of 1.5 kN/m and is also carried at the upper chord joints. If the diagonal members are designed to resist tension only, find which panels require counterbracing.

A family of influence lines may be drawn as shown in Fig. 20.19(b) for the shear force in each of the 10 panels. We begin the analysis at the centre of the truss where the DLS force has its least

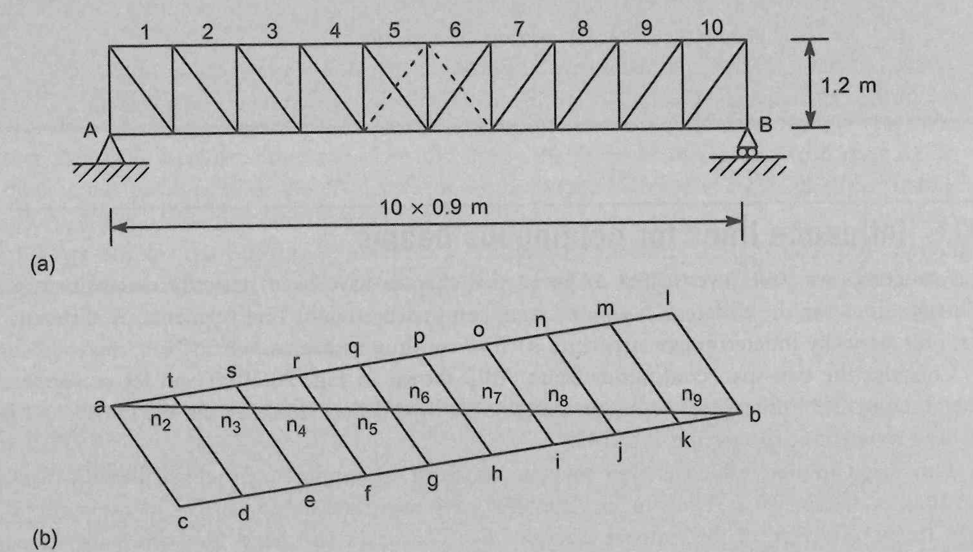


FIGURE 20.19

Counterbracing in a Pratt truss.

; initially, therefore, we consider panel 5. The shear force, S_5, in panel 5 with the head of the oad at n_5 is given by

$$S_5 = 1.0\,(\text{area } n_5qa - \text{area } n_5gb) + 1.5\,(\text{area } n_5qa)$$

$$S_5 = -1.0 \times \text{area } n_5gb + 2.5 \times \text{area } n_5qa \qquad \text{(i)}$$

ne ordinates in the S_5 influence line at g and q are found from similar triangles and are 0.5 and espectively. Also, from similar triangles, n_5 divides the horizontal distance between q and g in atio 0.4:0.5. Therefore, from Eq. (i)

$$S_5 = -1.0 \times \frac{1}{2} \times 5.0 \times 0.5 + 2.5 \times \frac{1}{2} \times 4.0 \times 0.4$$

n gives

$$S_5 = 0.75 \text{ kN}$$

nerefore, since S_5 is positive, the diagonal in panel 5 will be in compression so that panel 5, and symmetry panel 6, requires counterbracing.

ow with the head of the live load at n_4, $S_4 = 1.0$ (area n_4ra − area n_4fb) + 1.5 (area n_4ra).

ne ordinates and base lengths in the triangles n_4fb and n_4ra are determined as before.

nen

$$S_4 = -1.0 \times \frac{1}{2} \times 6.0 \times 0.6 + 2.5 \times \frac{1}{2} \times 3.0 \times 0.3$$

which

$$S_4 = -0.67 \text{ kN}$$

erefore, since S_4 is negative, panel 4, and therefore panel 7, do not require counterbracing.

early the remaining panels will not require counterbracing.

ote that for a Pratt truss having an odd number of panels the net value of the dead load shear in the central panel is zero, so that this panel will always require counterbracing.

Influence lines for continuous beams

uctures we have investigated so far in this chapter have been statically determinate so that the ce lines for the different functions have comprised straight line segments. A different situation or statically indeterminate structures such as continuous beams.

nsider the two-span continuous beam ABC shown in Fig. 20.20(a) and let us suppose that we construct influence lines for the reaction at B, the shear force at the section D in AB and the g moment at the section F in BC.

shape of the influence lines may be obtained by employing the Mueller–Breslau principle ed in Section 20.2. Thus, in Fig. 20.20(b) we remove the support at B and apply a unit displace- n the direction of the support reaction, R_B. The beam will bend into the shape shown since it pinned to the supports at A and C.

s would not have been the case, of course, if the span BC did not exist for then the beam would bout A as a rigid link and the R_B influence line would have been straight as in Fig. 20.1(c).

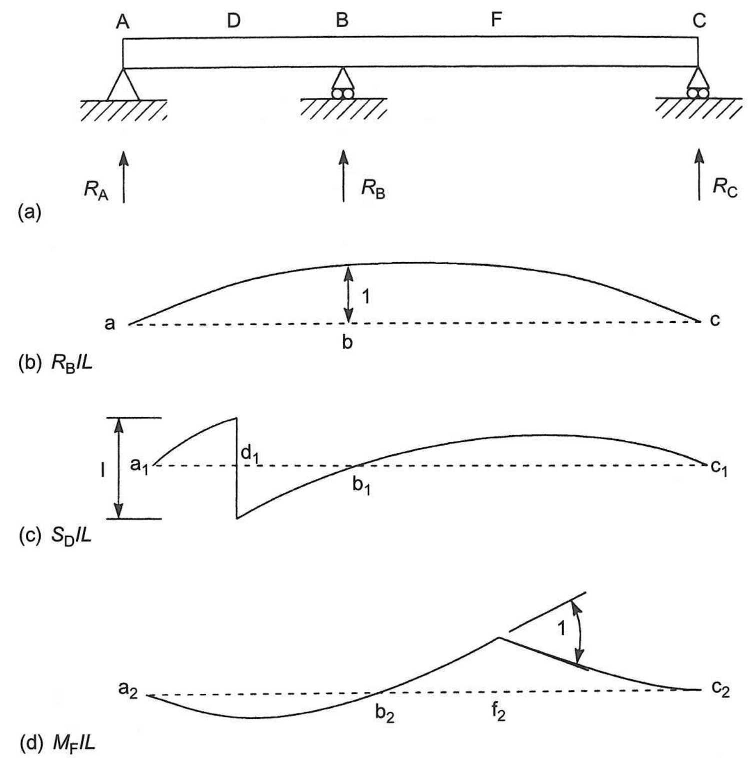


FIGURE 20.20

Influence lines for a continuous beam using the Mueller–Breslau principle.

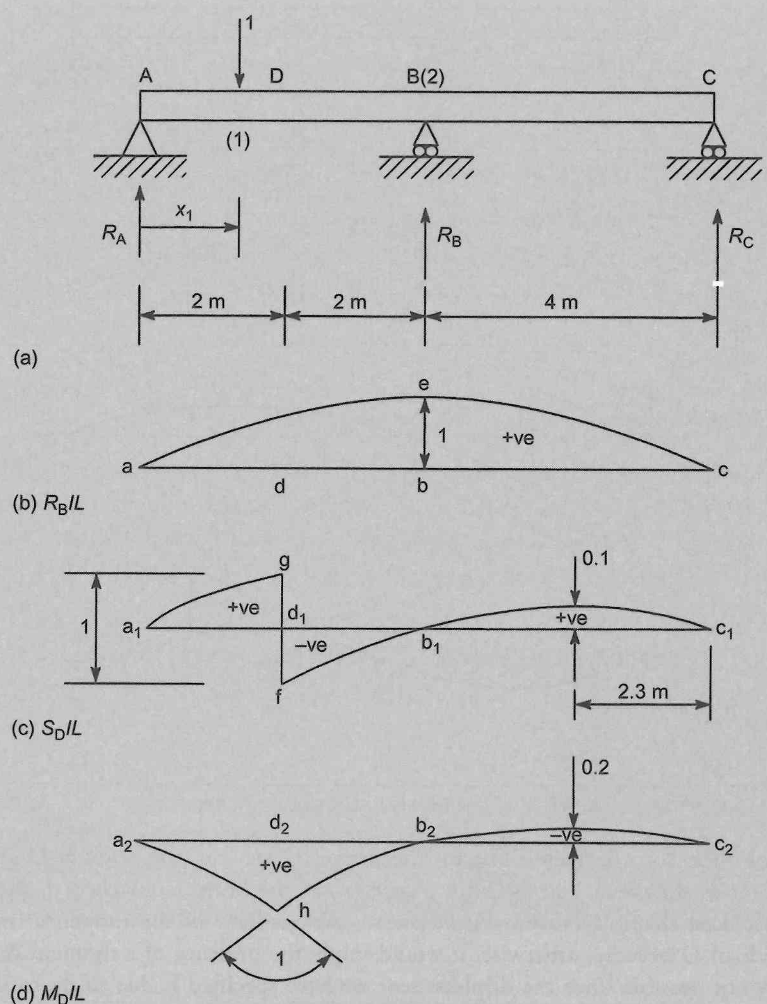
To obtain the shear force influence line for the section D we 'cut' the beam at D and apply a unit shear displacement as shown in Fig. 20.20(c). Again, since the beam is attached to the support at C, the resulting displaced shape is curved. Furthermore, the gradient of the influence line must be the same on each side of D because, otherwise, it would imply the presence of a moment causing a relative rotation. This is not possible since the displacement we have specified is due solely to shear. It follows that the influence line between A and D must also be curved.

The influence line for the bending moment at F is found by inserting a hinge at F and applying a relative unit rotation as shown in Fig. 20.20(d). Again the portion ABF of the beam will be curved, as will the portion FC, since this part of the beam must rotate so that the sum of the rotations of the two portions of the beam at F is equal to unity.

EXAMPLE 20.9

Construct influence lines for the reaction at B and for the shear force and bending moment at D in the two-span continuous beam shown in Fig. 20.21(a).

The shape of each influence line may be drawn using the Mueller–Breslau principle as shown in Fig. 20.21(b), (c) and (d). However, before they can be of direct use in determining maximum values, say, of the various functions due to the passage of loading systems, the ordinates must be calculated; for this, since the influence lines are comprised of curved segments, we need to derive their equations.



RE 20.21

ence lines for the continuous beam of Ex. 20.9.

wever, once the influence line for a support reaction, R_B in this case, has been established, the
ting influence lines follow from statical equilibrium.

luence line

se initially that a unit load is a distance x_1 from A, between A and B. To determine R_B we may e flexibility method described in Section 16.4. Thus we remove the support at B (point 2) lculate the displacement, a_{21}, at B due to the unit load at x_1 (point 1). We then calculate the cement, a_{22}, at B due to a vertically downward unit load at B. The total displacement at B due unit load at x_1 and the reaction R_B is then

$$a_{21} - a_{22}R_B = 0 \quad \text{(i)}$$

since the support at B is not displaced. In Eq. (i) the term $a_{22}R_B$ is negative since R_B is in the opposite direction to the applied unit load at B.

Both the flexibility coefficients in Eq. (i) may be obtained from a single unit load application since, from the reciprocal theorem (Section 15.4), the displacement at B due to a unit load at x_1 is equal to the displacement at x_1 due to a unit load at B. Therefore we apply a vertically downward unit load at B.

The equation for the displaced shape of the beam is that for a simply supported beam carrying a central concentrated load. Therefore, from Eq. (iv) of Ex. 13.5

$$v = \frac{1}{48EI}(4x^3 - 3L^2x) \quad \text{(ii)}$$

or, for the beam of Fig. 20.21(a)

$$v = \frac{x}{12EI}(x^2 - 48) \quad \text{(iii)}$$

At B, when $x = 4$ m

$$v_B = -\frac{32}{3EI} = a_{22} \quad \text{(iv)}$$

Furthermore, the displacement at B due to the unit load at x_1 (=displacement at x_1 due to a unit load at B) is from Eq. (iii)

$$v_{x_1} = \frac{x_1}{12EI}(x_1^2 - 48) = a_{21} \quad \text{(v)}$$

Substituting for a_{22} and a_{21} in Eq. (i) we have

$$\frac{x_1}{12EI}(x_1^2 - 48) + \frac{32}{3EI}R_B = 0$$

from which

$$R_B = -\frac{x_1}{128}(x_1^2 - 48) \quad (0 \le x_1 \le 4.0\text{ m}) \quad \text{(vi)}$$

Equation (vi) gives the influence line for R_B with the unit load between A and B; the remainder of the influence line follows from symmetry. Eq. (vi) may be checked since we know the value of R_B with the unit load at A and B. Thus from Eq. (vi), when $x_1 = 0$, $R_B = 0$ and when $x_1 = 4.0$ m, $R_B = 1$ as expected.

If the support at B were not symmetrically positioned, the above procedure would be repeated for the unit load on the span BC. In this case the equations for the deflected shape of AB and BC would be Eqs (xiv) and (xv) in Ex. 13.6.

In this example we require the S_D influence line so that we shall, in fact, need to consider the value of R_B with the unit load on the span BC. Therefore from Eq. (xv) in Ex. 13.6

$$v_{x_1} = -\frac{1}{12EI}(x_1^3 - 24x_1^2 + 144x_1 - 128) \quad (4.0\text{ m} \le x_1 \le 8.0\text{ m}) \quad \text{(vii)}$$

ıce from Eq. (i)

$$R_B = \frac{1}{128}(x_1^3 - 24x_1^2 + 144x_1 - 128) \quad (4.0\text{ m} \le x_1 \le 8.0\text{ m}) \tag{viii}$$

ıeck on Eq. (viii) shows that when $x_1 = 4.0$ m, $R_B = 1$ and when $x_1 = 8.0$ m, $R_B = 0$.

uence line

he unit load to the left of D, the shear force, S_D, at D is most simply given by

$$S_D = -R_A + 1 \tag{ix}$$

by taking moments about C, we have

$$R_A \times 8 - 1(8 - x_1) + R_B \times 4 = 0 \tag{x}$$

stituting in Eq. (x) for R_B from Eq. (vi) and rearranging gives

$$R_A = \frac{1}{256}(x_1^3 - 80x_1 + 256) \tag{xi}$$

:, from Eq. (ix)

$$S_D = -\frac{1}{256}(x_1^3 - 80x_1) \quad (0 \le x_1 \le 2.0\text{ m}) \tag{xii}$$

ɪrefore, when $x_1 = 0$, $S_D = 0$ and when $x_1 = 2.0$ m, $S_D = 0.59$, the ordinate d_1g in the S_D ce line in Fig. 20.21(c).

h the unit load between D and B

$$S_D = -R_A$$

, substituting for R_A from Eq. (xi)

$$S_D = -\frac{1}{256}(x_1^3 - 80x_1 + 256) \quad (2.0\text{ m} \le x_1 \le 4.0\text{ m}) \tag{xiii}$$

ıs, when $x_1 = 2.0$ m, $S_D = -0.41$, the ordinate d_1f in Fig. 20.21(c) and when $x_1 = 4.0$ m,

.

v consider the unit load between B and C. Again

$$S_D = -R_A$$

this case, R_B in Eq. (x) is given by Eq. (viii). Substituting for R_B from Eq. (viii) in Eq. (x) we

$$R_A = -S_D = -\frac{1}{256}(x_1^3 - 24x_1^2 + 176x_1 - 384) \quad (4.0\text{ m} \le x_1 \le 8.0\text{ m}) \tag{xiv}$$

:refore the S_D influence line consists of three segments, a_1g, fb_1 and b_1c_1.

luence line

he unit load between A and D

$$M_D = R_A \times 2 + 1(2 - x_1) \tag{xv}$$

stituting for R_A from Eq. (xi) in Eq. (xv) and simplifying, we obtain

$$M_D = \frac{1}{128}(x_1^3 - 48x_1) \quad (0 \le x_1 \le 2.0\text{ m}) \tag{xvi}$$

When $x_1 = 0$, $M_D = 0$ and when $x_1 = 2.0$ m, $M_D = 0.81$, the ordinate d_2h in the M_D influence line in Fig. 20.21(d).

Now with the unit load between D and B

$$M_D = R_A \times 2 \tag{xvii}$$

Therefore, substituting for R_A from Eq. (xi) we have

$$M_D = \frac{1}{128}(x_1^3 - 80x_1 + 256) \quad (2.0\text{ m} \le x_1 \le 4.0\text{ m}) \tag{xviii}$$

From Eq. (xviii) we see that when $x_1 = 2.0$ m, $M_D = 0.81$, again the ordinate d_2h in Fig. 20.21(d). Also, when $x_1 = 4.0$ m, $M_D = 0$.

Finally, with the unit load between B and C, M_D is again given by Eq. (xvii) but in which R_A is given by Eq. (xiv). Hence

$$M_D = -\frac{1}{128}(x_1^3 - 24x_1^2 + 176x_1 - 384) \quad (4.0\text{ m} \le x_1 \le 8.0\text{ m}) \tag{xix}$$

The maximum ordinates in the S_D and M_D influence lines for the span BC may be found by differentiating Eqs (xiv) and (xix) with respect to x_1, equating to zero and then substituting the resulting values of x_1 back in the equations. Thus, for example, from Eq. (xiv)

$$\frac{dS_D}{dx_1} = \frac{1}{256}(3x_1^2 - 48x_1 + 176) = 0$$

from which $x_1 = 5.7$ m. Hence

$$S_D(\max) = 0.1$$

Similarly $M_D(\max) = -0.2$ at $x_1 = 5.7$ m.

In this chapter we have constructed influence lines for beams, trusses and continuous beams. Clearly influence lines can be drawn for a wide variety of structures that carry moving loads. Their construction, whatever the structure, is based on considering the passage of a unit load across the structure.

PROBLEMS

P.20.1 Construct influence lines for the support reaction at A in the beams shown in Fig. P.20.1(a), (b) and (c).

Ans.

a. Unit load at C, $R_A = 1.25$.

b. Unit load at C, $R_A = 1.25$; at D, $R_A = -0.25$.

c. Unit load between A and B, $R_A = 1$; at C, $R_A = 0$

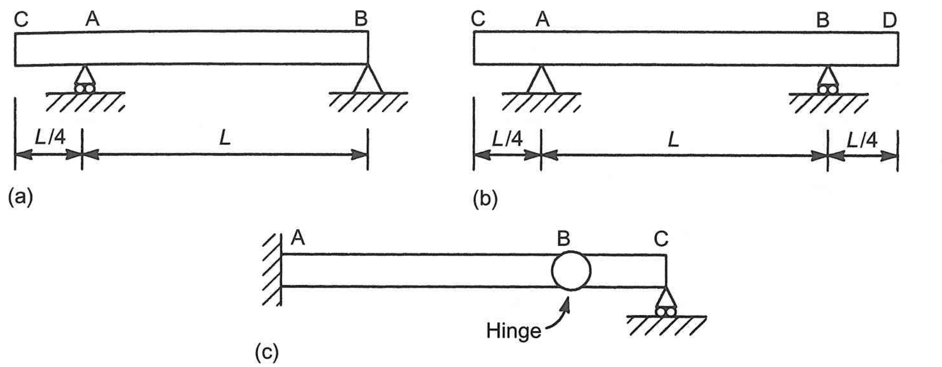


FIGURE P.20.1

Draw influence lines for the shear force at C in the beams shown in Fig. P.20.2(a) and (b).

Ans. Influence line ordinates

a. D = −0.25, A = 0, C = ± 0.5, B = 0.
b. D = −0.25, A = B = 0, C = ± 0.5, E = 0.25.

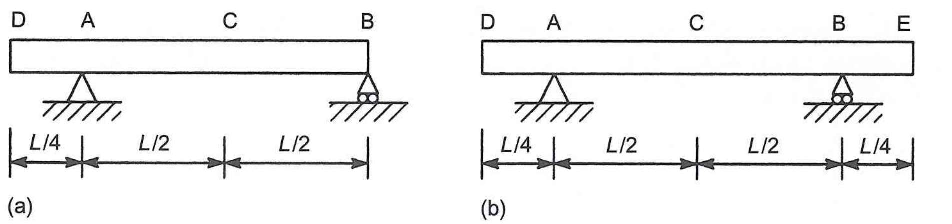


FIGURE P.20.2

Draw influence lines for the bending moment at C in the beams shown in Fig. P.20.2(a) and (b).

Ans. Influence line ordinates

a. D = −0.125*L*, A = B = 0, C = 0.25*L*.
b. D = E = −0.125*L*, A = B = 0, C = 0.25*L*.

The simply supported beam shown in Fig. P.20.4 carries a uniformly distributed travelling load of length 10 m and intensity 20 kN/m. Calculate the maximum positive and negative values of shear force and bending moment at the section C of the beam.

Ans. S_C = − 37.5 kN, +40.0 kN M_C = +550 kN m, −80 kN m.

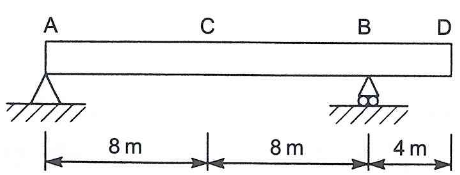


FIGURE P.20.4

P.20.5 The beam shown in Fig. P.20.5(a) is crossed by the train of four loads shown in Fig. P.20.5(b). For a section at mid-span, determine the maximum sagging and hogging bending moments.

Ans. +161.3 kN m, −77.5 kN m.

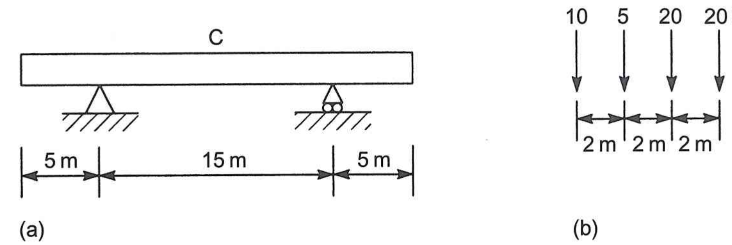


FIGURE P.20.5

P.20.6 A simply supported beam AB of span 20 m is crossed by the train of loads shown in Fig. P.20.6. Determine the position and magnitude of the absolute maximum bending moment on the beam and also the maximum values of positive and negative shear force anywhere on the beam.

Ans. *M* (max) = 466.7 kN m under a central load 10.5 m from A.
S(max −ve) = −104 kN at A, *S*(max +ve) = 97.5 kN at B.

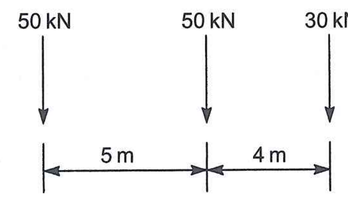


FIGURE P.20.6

P.20.7 The three-span beam shown in Fig. P.20.7 has hinges at C and E in its central span. Construct influence lines for the reaction at B and for the shear force and bending moment at the sections K and D.

Ans. Influence line ordinates
R_B; A = 0, B = 1, C = 1.25, E = F = G = 0.
S_K; A = 0, K = ± 0.5, B = 0, C = +0.25, E = 0.
S_D; A = B = 0, D = −1.0, C = −1.0, E = F = G = 0.
M_K; A = B = 0, K = 1.0, C = −0.5, E = F = G = 0.
M_D; A = B = D = 0, C = −0.5, E = F = G = 0.

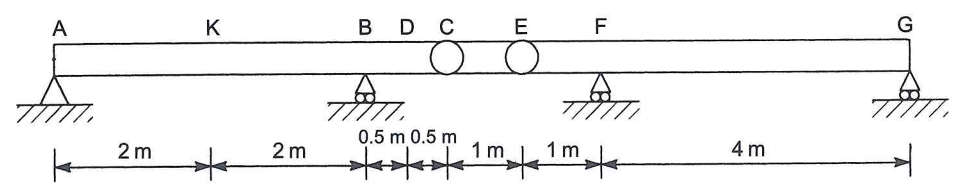


FIGURE P.20.7

Draw influence lines for the reactions at A and C and for the bending moment at E in the beam system shown in Fig. P.20.8. Note that the beam AB is supported on the lower beam at D by a roller.

If two 10 kN loads, 5 m apart, cross the upper beam AB, determine the maximum values of the reactions at A and C and the bending moment at E.

Ans. R_A(max) = 16.7 kN, R_C(max) = 17.5 kN, M_E(max) = 58.3 kN m.

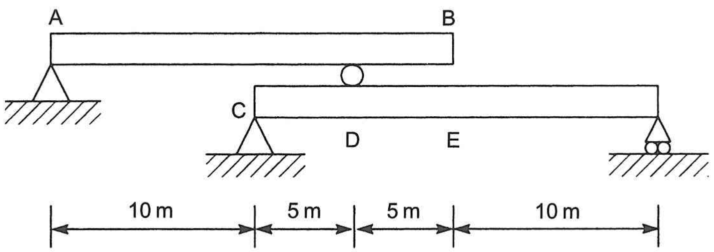


FIGURE P.20.8

A simply supported beam having a span of 5 m has a self-weight of 0.5 kN/m and carries a travelling uniformly distributed load of intensity 1.2 kN/m and length 1 m. Calculate the length of beam over which shear reversal occurs.

Ans. The central 1.3 m (graphical solution).

0 Construct an influence line for the force in the member CD of the truss shown in Fig. P.20.10 and calculate the force in the member produced by the loads positioned at C, D and E.

Ans. 28.1 kN (compression).

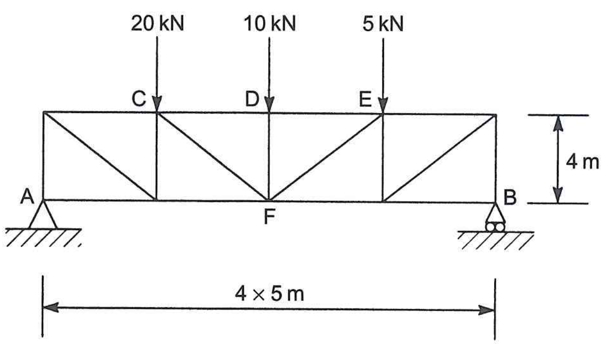


FIGURE P.20.10

1 The truss shown in Fig. P.20.11 carries a train of loads consisting of, left to right, 40, 70, 70 and 60 kN spaced at 2, 3 and 3 m, respectively. If the self-weight of the truss is 15 kN/m, calculate the maximum force in each of the members CG, HD and FE.

Ans. CG = 763 kN, HD = −724 kN, FE = −307 kN.

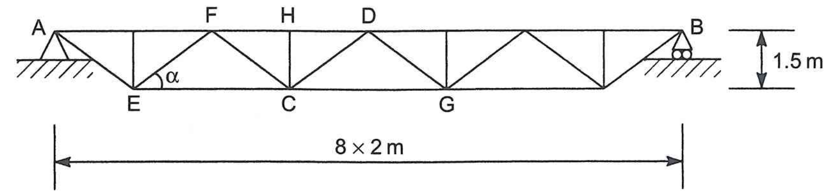


FIGURE P.20.11

P.20.12 One of the main girders of a bridge is the truss shown in Fig. P.20.12. Loads are transmitted to the truss through cross beams attached at the lower panel points. The self-weight of the truss is 30 kN/m and it carries a live load of intensity 15 kN/m and of length greater than the span. Draw influence lines for the force in each of the members CE and DE and determine their maximum values.

Ans. CE = +37.3 kN, −65.3 kN, DE = +961.2 kN.

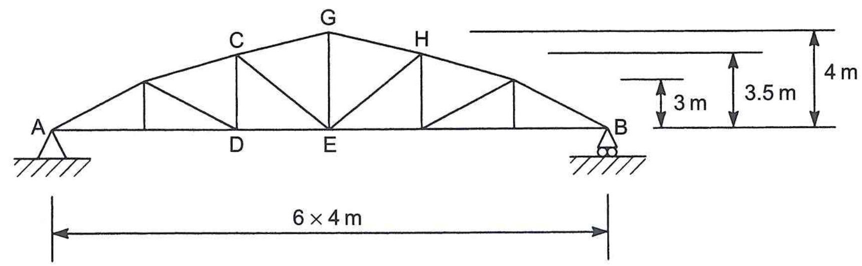


FIGURE P.20.12

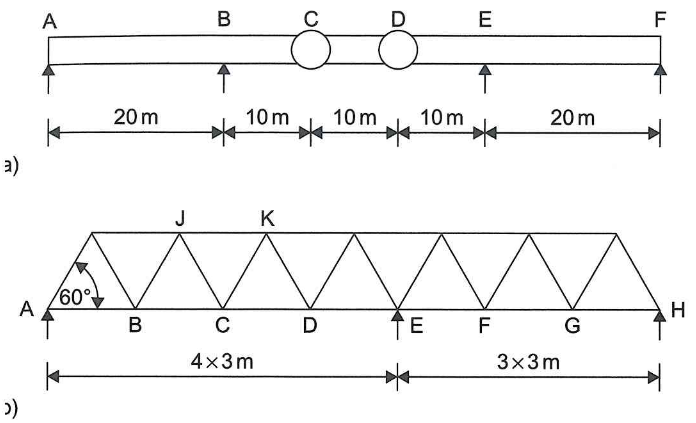
P.20.13 Fig. P.20.13(a) represents a bridge structure with a suspended span so that C and D are hinged connections. Sketch dimensioned influence lines for the vertical reactions at A and B and the shearing force at the hinge C.

The truss shown in Fig. P.20.13(b) is supported at A, E and H. A scale model of the truss, supported at A and H only, was loaded by a vertical load at E which produced the following values of deflection:

A	B	C	D	E	F	G	H
0	10	16	21	27	20	13	0

Plot the unit influence line for the vertical reaction at E in the real truss and hence find its value when concentrated loads of 120 kN at B and 160 kN at C are applied. For this loading system find the values of the axial force in the members JC and JK.

Ans. See Solutions Manual for influence lines. R_E = 139.3 kN.



a)

b)

FIGURE P.20.13

4 The Pratt truss shown in Fig. P.20.14 has a self-weight of 1.2 kN/m and carries a uniformly distributed live load longer than the span of intensity 2.8 kN/m, both being applied at the upper chord joints. If the diagonal members are designed to resist tension only, determine which panels require counterbracing.

Ans. Panels 4, 5 and 6.

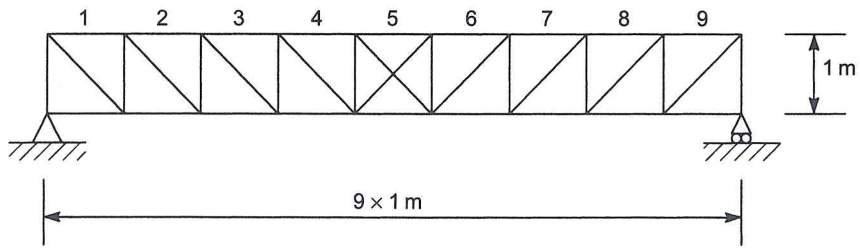


FIGURE P.20.14

5 Using the Mueller–Breslau principle sketch the shape of the influence lines for the support reactions at A and B, and the shear force and bending moment at E in the continuous beam shown in Fig. P.20.15.

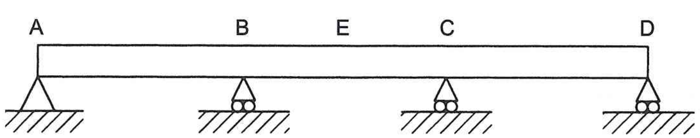


FIGURE P.20.15

5 Determine the equation of the influence line for the reaction at A in the continuous beam shown in Fig. P.20.16 and determine its value when a load of 30 kN/m covers the span AB.

Ans.

$$R_A = \frac{3}{16}\left\{\frac{x^3}{6} - \frac{1}{3}[x-2]^3 - \frac{10}{3}x + \frac{16}{3}\right\} \quad 26.25\ \text{kN.}$$

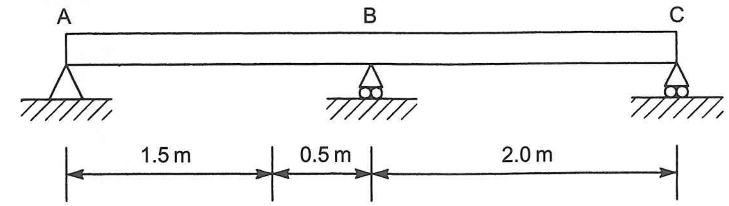


FIGURE P.20.16

P.20.17 Derive the equation for the unit influence line for the reaction at the support B in the continuous beam shown in Fig. P.20.17 and calculate its value if a uniformly distributed load of intensity 20 kN/m covers the span AB; the flexural rigidity of the beam is *EI*. Also sketch the unit influence lines for the shear force and bending moment at the point D.

Ans.

$$y = \frac{3}{32}\left\{4x + \frac{1}{6}[x-4]^3 - \frac{x^3}{12}\right\}, \quad R_B = 50\ \text{kN.}$$

See Solutions Manual for I.Ls.

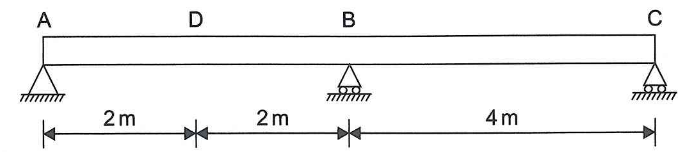


FIGURE P.20.17

P.20.18 Derive the equation for the unit influence line for the vertical reaction at B in the continuous beam shown in Fig. P.20.18 and hence find its value when a uniformly distributed load of intensity 24 kN/m covers the span AB.

Ans.

$$y = \frac{1}{96}\left\{20x + \frac{1}{6}[x-6]^3 - \frac{x^3}{9}\right\}, \quad R_B = 81\ \text{kN.}$$

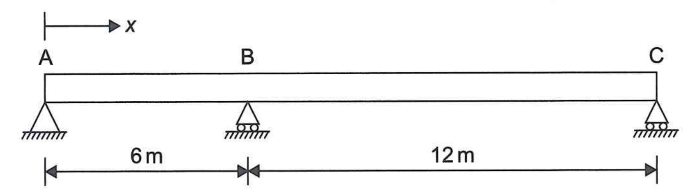


FIGURE P.20.18

CHAPTER

uctural Instability

21

in considering the behaviour of structural members under load, we have been concerned with ɔility to withstand different forms of stress. Their strength, therefore, has depended upon the h properties of the material from which they are fabricated. However, structural members sub- :o axial compressive loads may fail in a manner that depends upon their geometrical properties than their material properties. It is common experience, for example, that a long slender struc- ıember such as that shown in Fig. 21.1(a) will suddenly bow with large lateral displacements ubjected to an axial compressive load (Fig. 21.1(b)). This phenomenon is known as *instability* e member is said to *buckle.* If the member is exceptionally long and slender it may regain its straight shape when the load is removed.

ıctural members subjected to axial compressive loads are known as *columns* or *struts*, although mer term is usually applied to the relatively heavy vertical members that are used to support and slabs; struts are compression members in frames and trusses.

clear from the above discussion that the design of compression members must take into account ly the material strength of the member but also its stability against buckling. Obviously the a member is in relation to its cross-sectional dimensions, the more likely it is that failure will be e in compression of the material rather than one due to instability. It follows that in some inter- e range a failure will be a combination of both.

shall investigate the buckling of long slender columns and derive expressions for the *buckling* or *load;* the discussion will then be extended to the design of columns of any length and to a con- on of beams subjected to axial load and bending moment.

Euler theory for slender columns

st significant contribution to the theory of the buckling of columns was made in the 18th cen- Euler. His classical approach is still valid for long slender columns possessing a variety of end ts. Before presenting the theory, however, we shall investigate the nature of buckling and the dif- between theory and practice.

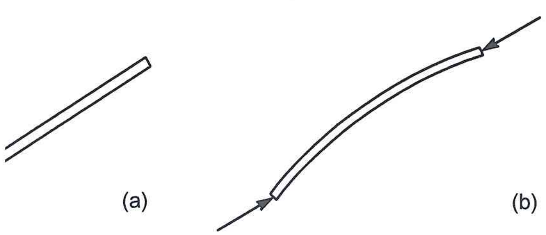


FIGURE 21.1

Buckling of a slender column.

We have seen that if an increasing axial compressive load is applied to a long slender column there is a value of load at which the column will suddenly bow or buckle in some unpredetermined direction. This load is patently the buckling load of the column or something very close to the buckling load. The fact that the column buckles in a particular direction implies a degree of asymmetry in the plane of the buckle caused by geometrical and/or material imperfections of the column and its load. Theoretically, however, in our analysis we stipulate a perfectly straight, homogeneous column in which the load is applied precisely along the perfectly straight centroidal axis. Theoretically, therefore, there can be no sudden bowing or buckling, only axial compression. Thus we require a precise definition of buckling load which may be used in the analysis of the perfect column.

If the perfect column of Fig. 21.2 is subjected to a compressive load P, only shortening of the column occurs no matter what the value of P. Clearly if P were to produce a stress greater than the yield stress of the material of the column, then material failure would occur. However, if the column is displaced a small amount by a lateral load, F, then, at values of P below the critical or buckling load, P_{CR}, removal of F results in a return of the column to its undisturbed position, indicating a state of stable equilibrium. When $P = P_{CR}$ the displacement does not disappear and the column will, in fact, remain in *any* displaced position so long as the displacement is small. Thus the buckling load, P_{CR}, is associated with a state of *neutral equilibrium. For* $P > P_{CR}$ enforced lateral displacements increase and the column is unstable.

Buckling load for a pin-ended column

Consider the pin-ended column shown in Fig. 21.3. We shall assume that it is in the displaced state of neutral equilibrium associated with buckling so that the compressive axial load has reached the value P_{CR}. We also assume that the column has deflected so that its displacements, v, referred to the axes $0xy$ are positive. The bending moment, M, at any section X is then given by

$$M = -P_{CR}v$$

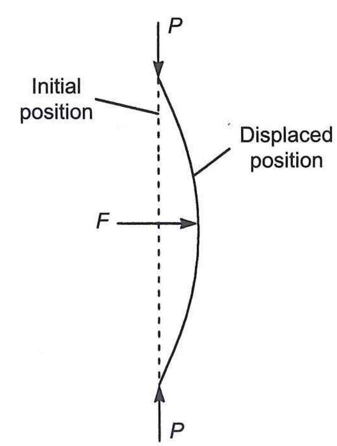


FIGURE 21.2

Definition of buckling load of a column.

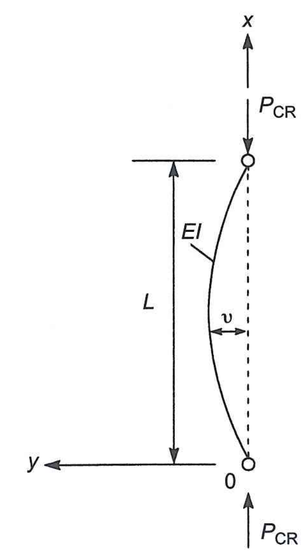


FIGURE 21.3

Determination of buckling load for a pin-ended substituting for M from Eq. (13.3) we obtain

$$\frac{d^2v}{dx^2} = -\frac{P_{CR}}{EI}v \tag{21.1}$$

rranging we obtain

$$\frac{d^2v}{dx^2} + \frac{P_{CR}}{EI}v = 0 \tag{21.2}$$

solution of Eq. (21.2) is of standard form and is

$$v = C_1 \cos \mu x + C_2 \sin \mu x \tag{21.3}$$

h C_1 and C_2 are arbitrary constants and $\mu^2 = P_{CR}/EI$. The boundary conditions for this particular $v = 0$ at $x = 0$ and $x = L$. The first of these gives $C_1 = 0$ while from the second we have

$$0 = C_2 \sin \mu L$$

a non-trivial solution (i.e. $v \neq 0$ and $C_2 \neq 0$) then

$$\sin \mu L = 0$$

$\mu L = n\pi$ where $n = 1, 2, 3, \ldots$
ce

$$\frac{P_{CR}}{EI}L^2 = n^2\pi^2$$

ich

$$P_{CR} = \frac{n^2\pi^2 EI}{L^2} \tag{21.4}$$

e that C_2 is indeterminate and that the displacement of the column cannot therefore be found. o be expected since the column is in neutral equilibrium in its buckled state.
smallest value of buckling load corresponds to a value of $n = 1$ in Eq. (21.4), i.e.

$$P_{CR} = \frac{\pi^2 EI}{L^2} \tag{21.5}$$

column then has the displaced shape $v = C_2 \sin \mu x$ and buckles into the longitudinal half sine-own in Fig. 21.4(a). Other values of P_{CR} corresponding to $n = 2, 3, \ldots$ are

$$P_{CR} = \frac{4\pi^2 EI}{L^2} \qquad P_{CR} = \frac{9\pi^2 EI}{L^2}, \ldots$$

e higher values of buckling load correspond to more complex buckling modes as shown in .4(b) and (c). Theoretically these different modes could be produced by applying external s to a slender column at the points of contraflexure to prevent lateral movement. However, in the lowest value is never exceeded since high stresses develop at this load and failure of the col-sues. Therefore we are not concerned with buckling loads higher than this.

ng load for a column with fixed ends

ice, columns usually have their ends restrained against rotation so that they are, in effect, fixed.

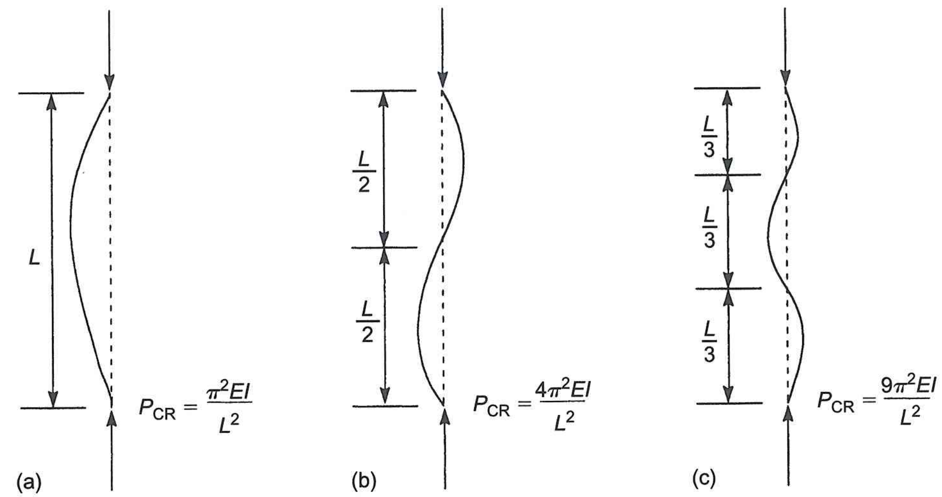


FIGURE 21.4

Buckling modes of a pin-ended column.

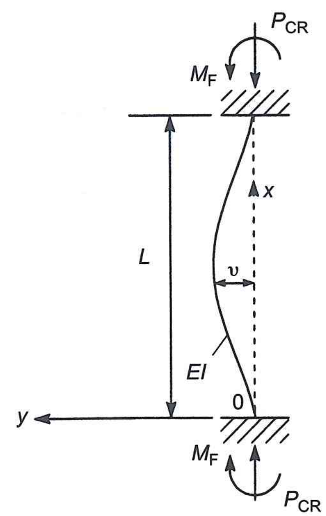


FIGURE 21.5

Buckling of a slender column with fixed ends.

reached the critical value, P_{CR}, so that the column is in a state of neutral equilibrium. In this case, the ends of the column are subjected to fixing moments, M_F, in addition to axial load. Thus at any section X the bending moment, M, is given by

$$M = -P_{CR}v + M_F$$

Substituting for M from Eq. (13.3) we have

$$\frac{d^2v}{dx^2} = -\frac{P_{CR}}{EI}v + \frac{M_F}{EI} \tag{21.6}$$

rranging we obtain

$$\frac{d^2v}{dx^2}a + \frac{P_{CR}}{EI}v = \frac{M_F}{EI} \qquad (21.7)$$

ıtion of which is

$$v = C_1\cos\mu x + C_2\sin\mu x + \frac{M_F}{P_{CR}} \qquad (21.8)$$

$$\mu^2 = \frac{P_{CR}}{EI}$$

en $x=0$, $v=0$ so that $C_1 = -M_F/P_{CR}$. Further $v=0$ at $x=L$, hence

$$0 = -\frac{M_F}{P_{CR}}\cos\mu L + C_2\sin\mu L + \frac{M_F}{P_{CR}}$$

;ives

$$C_2 = -\frac{M_F}{P_{CR}}\frac{(1-\cos\mu L)}{\sin\mu L}$$

ıce Eq. (21.8) becomes

$$v = \frac{M_F}{P_{CR}}\left[\cos\mu x + \frac{(1-\cos\mu L)}{\sin\mu L}\sin\mu x - 1\right] \qquad (21.9)$$

e that again v is indeterminate since M_F cannot be found. Also since $dv/dx=0$ at $x=L$ we ›m Eq. (21.9)

$$0 = 1 - \cos\mu L$$

$$\cos\mu L = 1$$

$$\mu L = n\pi \quad \text{where } n = 0, 2, 4, \cdots$$

a non-trivial solution, i.e. $n \neq 0$, and taking the smallest value of buckling load ($n=2$) we have

$$P_{CR} = \frac{4\pi^2 EI}{L^2} \qquad (21.10)$$

ng load for a column with one end fixed and one end free

:onfiguration the upper end of the column is free to move laterally and also to rotate as shown ?1.6. At any section X the bending moment M is given by

$$M = P_{CR}(\delta - v) \text{ or } M = -P_{CR}v + M_F$$

:tituting for M in the first of these expressions from Eq. (13.3) (equally we could use the sec- : obtain

$$\frac{d^2v}{dx^2} = \frac{P_{CR}}{EI}(\delta - v) \qquad (21.11)$$

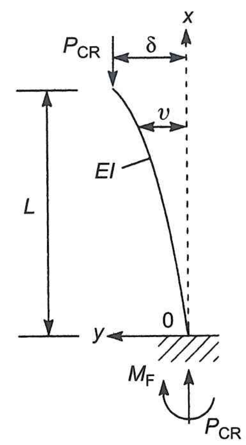


FIGURE 21.6

Determination of buckling load for a column with one end fixed and one end free.

which, on rearranging, becomes

$$\frac{d^2v}{dx^2} + \frac{P_{CR}}{EI}v = \frac{P_{CR}}{EI}\delta \qquad (21.12)$$

The solution of Eq. (21.12) is

$$v = C_1\cos\mu x + C_2\sin\mu x + \delta \qquad (21.13)$$

where $\mu^2 = P_{CR}/EI$. When $x=0$, $v=0$ so that $C_1 = -\delta$. Also when $x=L$, $v=\delta$ so that from Eq. (21.13) we have

$$\delta = -\delta\cos\mu L + C_2\sin\mu L + \delta$$

which gives

$$C_2 = \delta\frac{\cos\mu L}{\sin\mu L}$$

Hence

$$v = -\delta\left(\cos\mu x - \frac{\cos\mu L}{\sin\mu L}\sin\mu x - 1\right) \qquad (21.14)$$

Again v is indeterminate since δ cannot be determined. Finally we have $dv/dx=0$ at $x=0$. Hence from Eq. (21.14)

$$\cos\mu L = 0$$

whence

$$\mu L = n\frac{\pi}{2} \quad \text{where } n = 1, 3, 5, \cdots$$

Thus taking the smallest value of buckling load (corresponding to $n=1$) we obtain

$$P_{CR} = \frac{\pi^2 EI}{\ldots} \qquad (21.15)$$

g of a column with one end fixed and the other pinned

mn in this case is allowed to rotate at one end but requires a lateral force, F, to maintain its Fig. 21.7).

y section X the bending moment M is given by

$$M = -P_{CR}v - F(L - x)$$

tuting for M from Eq. (13.3) we have

$$\frac{d^2v}{dx^2} = -\frac{P_{CR}}{EI}v - \frac{F}{EI}(L - x) \tag{21.16}$$

ı rearranging, becomes

$$\frac{d^2v}{dx^2} + \frac{P_{CR}}{EI}v = -\frac{F}{EI}(L - x) \tag{21.17}$$

olution of Eq. (21.17) is

$$v = C_1\cos\mu x + C_2\sin\mu x - \frac{F}{P_{CR}}(L - x) \tag{21.18}$$

lv/dx = 0 at $x = 0$, so that

$$0 = \mu C_2 + \frac{F}{P_{CR}}$$

:h

$$C_2 = -\frac{F}{\mu P_{CR}}$$

$x = L$, $v = 0$, hence

$$0 = C_1\cos\mu L + C_2\sin\mu L$$

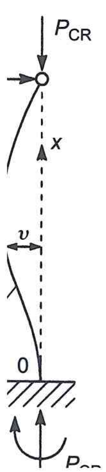


FIGURE 21.7

Determination of buckling load for a column with one end fixed and the other end

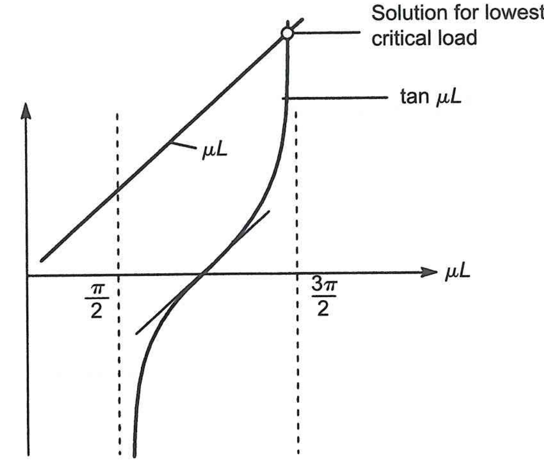


FIGURE 21.8

Solution of a transcendental equation.

which gives

$$C_1 = \frac{F}{\mu P_{CR}}\tan\mu L$$

Thus Eq. (21.18) becomes

$$v = \frac{F}{\mu P_{CR}}[\tan\mu L\cos\mu x - \sin\mu x - \mu(L - x)] \tag{21.19}$$

Also $v = 0$ at $x = 0$. Then

$$0 = \tan\mu L - \mu L$$

or

$$\mu L = \tan\mu L \tag{21.20}$$

Equation (21.20) is a transcendental equation which may be solved graphically as shown in Fig. 21.8. The smallest non-zero value satisfying Eq. (21.20) is approximately 4.49.

This gives

$$P_{CR} = \frac{20.2EI}{L^2}$$

which may be written approximately as

$$P_{CR} = \frac{2.05\pi^2 EI}{L^2} \tag{21.21}$$

It can be seen from Eqs (21.5), (21.10), (21.15) and (21.21) that the buckling load in all cases has the form

$$P_{CR} = \frac{K^2\pi^2 EI}{L^2} \tag{21.22}$$

in which K is some constant. Equation (21.22) may be written in the form

$$P_{CR} = \frac{\pi^2 EI}{\ldots} \tag{21.23}$$

ı $L_e(=L/K)$ is the *equivalent length* of the column, i.e. (by comparison of Eqs (21.23) and he length of a pin-ended column that has the same buckling load as the actual column. Clearly :ling load of any column may be expressed in this form so long as its equivalent length is By inspection of Eqs (21.5), (21.10), (21.15) and (21.21) we see that the equivalent lengths of us types of column are

both ends pinned	$L_e = 1.0\,L$
both ends fixed	$L_e = 0.5\,L$
one end fixed and one free	$L_e = 2.0\,L$
one end fixed and one pinned	$L_e = 0.7\,L$

PLE 21.1

ırm column of length L and flexural stiffness EI is simply supported at its ends and has an ıal elastic support at mid-span. This support is such that a lateral displacement v_c causes a g force kv_c to be generated at the point. Derive an expression for the buckling load of the col- the buckling load is $4\pi^2 EI/L^2$ find the value of k. Also, if the elastic support is infinitely stiff ıat the buckling load is given by the equation $\tan\lambda L = \lambda L/2$ where $\lambda = \sqrt{(P/EI)}$
column is shown in its displaced position in Fig. 21.9. The bending moment at any section olumn is given by

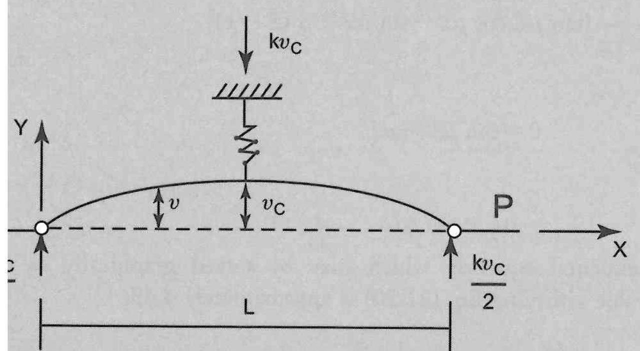


FIGURE 21.9
Column of Ex. 21.1

$$M = Pv - \frac{kv_c}{2}x$$

oy comparison with Eq. (21.1)

$$EI\frac{d^2v}{dx^2} = -Pv + \frac{kv_c}{2}x$$

$$\frac{d^2v}{dx^2} + \lambda^2 v = \frac{kv_c}{2EI}x \qquad \text{(i)}$$

$^2 = P/EI.$

The solution of Eq. (i) is

$$v = A\cos\lambda x + B\sin\lambda x + \frac{kv_c}{2P}x$$

The boundary conditions are $v=0$ when $x=0$, $v=v_c$ when $x=L/2$ and $(dv/dx)=0$ when $x=L/2$. From the first of these $A=0$ while from the second

$$B = v_c(1 - kL/4P)/\sin(\lambda L/2)$$

The third boundary condition gives, since $v_c \neq 0$

$$\left(1-\frac{kL}{4P}\right)\cos\frac{\lambda L}{2} + \frac{k}{2P\lambda}\sin\frac{\lambda L}{2} = 0$$

Rearranging

$$P = \frac{kL}{4P}\left(1 - \frac{\tan\lambda L/2}{\lambda L/2}\right)$$

If the buckling load $P = 4\pi^2 EI/L^2$ then $\lambda L/2 = \pi$ so that $k = 4P/L$. Finally, if $k \to \infty$

$$\frac{\tan\lambda L}{2} = \frac{\lambda L}{2} \qquad \text{(ii)}$$

Note that Eq. (ii) is the transcendental equation which would be derived when determining the buckling load of a column of length $L/2$, built in at one end and pinned at the other (see Eq. (21.23).

21.2 Limitations of the Euler theory

For a column of cross-sectional area A the critical stress, σ_{CR}, is, from Eq. (21.23)

$$\sigma_{CR} = \frac{P_{CR}}{A} = \frac{\pi^2 EI}{AL_e^2} \qquad (21.24)$$

The second moment of area, I, of the cross section is equal to Ar^2 where r is the *radius of gyration* of the cross section. Thus we may write Eq. (21.24) as

$$\sigma_{CR} = \frac{\pi^2 E}{(L_e/r)^2} \qquad (21.25)$$

Therefore for a column of a given material, the critical or buckling stress is inversely proportional to the parameter $(L_e/r)^2$. L_e/r is an expression of the proportions of the length and cross-sectional dimensions of the column and is known as its *slenderness ratio*. Clearly if the column is long and slender L_e/r is large and σ_{CR} is small; conversely, for a short column having a comparatively large area of cross section, L_e/r is small and σ_{CR} is high. A graph of σ_{CR} against L_e/r for a particular material has the form shown in

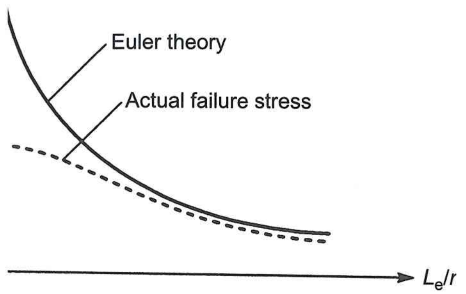


FIGURE 21.10

Variation of critical stress with slenderness ratio.

n compression rather than by buckling so that σ_{CR} as predicted by the Euler theory is no longer ıus in Fig. 21.10, the actual failure stress follows the dotted curve rather than the full line.

Failure of columns of any length

ıl or semi-empirical methods are generally used to predict the failure of a column of any length: n form the basis for safe load or safe stress tables given in Codes of Practice. One such method ves good agreement with experiment is that due to Rankine.

e theory

that P is the failure load of a column of a given material and of any length. Suppose also that P_S ure load in compression of a short column of the same material and that P_{CR} is the buckling load slender column, again of the same material. The Rankine theory proposes that

$$\frac{1}{P} = \frac{1}{P_S} + \frac{1}{P_{CR}} \tag{21.26}$$

tion (21.26) is valid for a very short column since $1/P_{CR} \to 0$ and P then $\to P_S$; the equation is for a long slender column since $1/P_S$ is small compared with $1/P_{CR}$; thus $P \to P_{CR}$. Therefore, :6) is seen to hold for extremes in column length.

let σ_S be the yield stress in compression of the material of the column and A its cross- area. Then

$$P_S = \sigma_S A$$

rom Eq. (21.23)

$$P_{CR} = \frac{\pi^2 EI}{L_e^2}$$

ituting for P_S and P_{CR} in Eq. (21.26) we have

$$\frac{1}{P} = \frac{1}{\sigma_S A} + \frac{1}{\pi^2 EI/L_e^2}$$

Thus

$$\frac{1}{P} = \frac{\pi^2 EI/L_e^2 + \sigma_S A}{\sigma_S A \pi^2 EI/L_e^2}$$

so that

$$P = \frac{\sigma_S A \pi^2 EI/L_e^2}{\pi^2 EI/L_e^2 + \sigma_S A}$$

Dividing top and bottom of the right-hand side of this equation by $\pi^2 EI/L_e^2$ we have

$$P = \frac{\sigma_S A}{1 + \sigma_S A L_e^2/\pi^2 EI}$$

But $I = Ar^2$ so that

$$P = \frac{\sigma_S A}{1 + (\sigma_S/\pi^2 E)(L_e/r)^2}$$

which may be written

$$P = \frac{\sigma_S A}{1 + k(L_e/r)^2} \tag{21.27}$$

in which k is a constant that depends upon the material of the column. The failure stress in compression, σ_C, of a column of any length is then, from Eq. (21.27)

$$\sigma_C = \frac{P}{A} = \frac{\sigma_S}{1 + k(L_e/r)^2} \tag{21.28}$$

Note that for a column of a given material σ_C is a function of the slenderness ratio, L_e/r.

EXAMPLE 21.2

A tubular column has an effective length of 2.5 m and is to be designed to carry a safe load of 300 kN. Assuming an approximate ratio of thickness to external diameter of 1/16 calculate a practical diameter and thickness using the Rankine formula with $\sigma_S = 330$ N/mm^2 and $k = 1/7500$. Use a factor of safety of 3.

The radius of gyration of the column is given by

$$r^2 = I/A$$

where I is the second moment of area of the column cross section and A its area. Then, if D is the external diameter

$$r^2 = \frac{(\pi/64)[D^4 - (7D/8)^4]}{(\pi/4)[D^2 - (7D/8)^2]}$$

which gives

$$r^2 = 0.11\,D^2$$

Substituting for r etc. in Eq. (21.28) and rearranging gives tion of which is

$$D = 122 \text{ mm}$$

the ratio of thickness to external diameter is 1:16 then a diameter of 122 mm would give ess of 7.6 mm. Therefore assume a thickness of 8 mm which gives an external diameter of .

curved column

ıtive approach to the Rankine theory bases a design formula on the failure of a column posses- ıll initial curvature, the argument being that in practice columns are never perfectly straight. ler the pin-ended column shown in Fig. 21.11. In its unloaded configuration the column has a al curvature such that the lateral displacement at any value of x is v_0. Let us assume that

$$v_0 = a \sin \pi \frac{x}{L} \tag{21.29}$$

ı is the initial displacement at the centre of the column. Equation (21.29) satisfies the bound- tions of $v_0 = 0$ at $x = 0$ and $x = L$ and also $dv_0/dx = 0$ at $x = L/2$; the assumed deflected ıerefore reasonable, particularly since we note that the buckled shape of a pin-ended column alf sine-wave.

the column is initially curved, an axial load, P, immediately produces bending and therefore eral displacements, v, measured from the initial displaced position. The bending moment, section X is then

$$M = -P(v + v_0) \tag{21.30}$$

olumn is initially unstressed, the bending moment at any section is proportional to the *change* in ıt that section from its initial configuration and not its absolute value. From Eq. (13.3)

$$M = EI \frac{d^2 v}{dx^2}$$

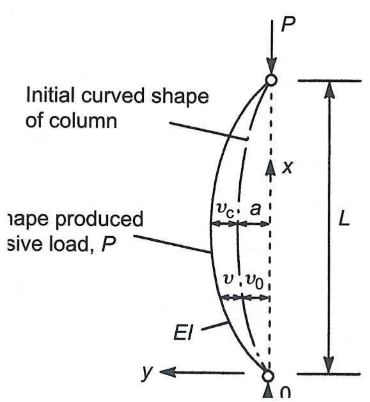


FIGURE 21.11

$$\frac{d^2 v}{dx^2} = -\frac{P}{EI}(v + v_0) \tag{21.31}$$

Rearranging Eq. (21.31) we have

$$\frac{d^2 v}{dx^2} + \frac{P}{EI} v = -\frac{P}{EI} v_0 \tag{21.32}$$

Note that P is not, in this case, the buckling load for the column. Substituting for v_0 from Eq. (21.29) we obtain

$$\frac{d^2 v}{dx^2} + \frac{P}{EI} v = -\frac{P}{EI} a \sin \pi \frac{x}{L} \tag{21.33}$$

The solution of Eq. (21.33) is

$$v = C_1 \cos \mu x + C_2 \sin \mu x + \frac{\mu^2 a}{(\pi^2/L^2) - \mu^2} \sin \pi \frac{x}{L} \tag{21.34}$$

in which $\mu^2 = P/EI$. If the ends of the column are pinned, $v = 0$ at $x = 0$ and $x = L$. The first of these boundary conditions gives $C_1 = 0$ while from the second we have

$$0 = C_2 \sin \mu L$$

Although this equation is identical to that derived from the boundary conditions of an initially straight, buckled, pin-ended column, the circumstances are now different. If $\sin \mu L = 0$ then $\mu L = \pi$ so that $\mu^2 = \pi^2/L^2$. This would then make the third term in Eq. (21.34) infinite which is clearly impossible for a column in stable equilibrium ($P < P_{CR}$). We conclude, therefore, that $C_2 = 0$ and hence Eq. (21.34) becomes

$$v = \frac{\mu^2 a}{(\pi^2/L^2) - \mu^2} \sin \pi \frac{x}{L} \tag{21.35}$$

Dividing the top and bottom of Eq. (21.35) by μ^2 we obtain

$$v = \frac{a \sin \pi x/L}{(\pi^2/\mu^2 L^2) - 1}$$

But $\mu^2 = P/EI$ and $a \sin \pi x/L = v_0$. Thus

$$v = \frac{v_0}{(\pi^2 EI/PL^2) - 1} \tag{21.36}$$

From Eq. (21.5) we see that $(\pi^2 EI/L^2 = P_{CR}$, the buckling load for a perfectly straight pin-ended column. Hence Eq. (21.36) becomes

$$v = \frac{v_0}{(P_{CR}/P) - 1} \tag{21.37}$$

It can be seen from Eq. (21.37) that the effect of the compressive load, P, is to increase the initial deflection, v_0, by a factor $1/[(P_{CR}/P) - 1]$. Clearly as P approaches P_{CR}, v tends to infinity. In practice this is impossible since material breakdown would occur before P_{CR} is reached.

If we consider displacements at the mid-height of the column we have from Eq. (21.37)

a

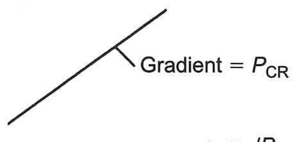


FIGURE 21.12

Experimental determination of the buckling load of a column from a Southwell plot.

ing we obtain

$$v_c = P_{CR}\frac{v_c}{P} - a \tag{21.38}$$

(21.38) represents a linear relationship between v_c and v_c/P. Thus in an actual test on an 'ed column a graph of v_c against v_c/P will be a straight line as the critical condition is The gradient of the line is P_{CR} and its intercept on the v_c axis is equal to a, the initial dis- : the mid-height of the column. The graph (Fig. 21.12) is known as a Southwell plot and :nient, non-destructive, method of determining the buckling load of columns.

imum bending moment in the column of Fig. 21.11 occurs at mid-height and is

$$M_{max} = -P(a + v_C)$$

ing for v_c from Eq. (21.38) we have

$$M_{max} = -Pa\left(1 + \frac{1}{(P_{CR}/P) - 1}\right)$$

$$M_{max} = -Pa\left(\frac{P_{CR}}{P_{CR} - P}\right) \tag{21.39}$$

imum compressive stress in the column occurs in an extreme fibre and is from Eq. (9.15)

$$\sigma_{max} = \frac{P}{A} + Pa\left(\frac{P_{CR}}{P_{CR} - P}\right)\left(\frac{c}{I}\right)$$

s the cross-sectional area, c is the distance from the centroidal axis to the extreme fibre and ıd moment of area of the column's cross section. Since $I = Ar^2$ (r = radius of gyration), we the above equation as

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{P_{CR}}{P_{CR} - P}\left(\frac{ac}{r^2}\right)\right) \tag{21.40}$$

! is the average stress, σ, on the cross section of the column. Thus, writing Eq. (21.40) in ss we have

$$\sigma_{max} = \sigma\left(1 + \frac{\sigma_{CR}}{\sigma_{CR} - \sigma}\left(\frac{ac}{r^2}\right)\right) \tag{21.41}$$

$_R = P_{CR}/A = \pi^2 E(r/L)^2$ (see Eq. (21.25)). The term ac/r^2 is an expression of the geometrical

$$\sigma_{max} = \sigma\left(1 + \frac{\eta\sigma_{CR}}{\sigma_{CR} - \sigma}\right) \tag{21.42}$$

Expanding Eq. (21.42) we have

$$\sigma_{max}(\sigma_{CR} - \sigma) = \sigma[(1 + \eta)\sigma_{CR} - \sigma]$$

which, on rearranging, becomes

$$\sigma^2 - \sigma[\sigma_{max} + (1 + \eta)\sigma_{CR}] + \sigma_{max}\sigma_{CR} = 0 \tag{21.43}$$

the solution of which is

$$\sigma = \frac{1}{2}[\sigma_{max} + (1 + \eta)\sigma_{CR}] - \sqrt{\frac{1}{4}[\sigma_{max} + (1 + \eta)\sigma_{CR}]^2 - \sigma_{max}\sigma_{CR}} \tag{21.44}$$

The positive square root in the solution of Eq. (21.43) is ignored since we are only interested in the smallest value of σ. Equation (21.44) then gives the average stress, σ, in the column at which the maximum compressive stress would be reached for any value of η. Thus if we specify the maximum stress to be equal to σ_Y, the yield stress of the material of the column, then Eq. (21.44) may be written

$$\sigma = \frac{1}{2}[\sigma_Y + (1 + \eta)\sigma_{CR}] - \sqrt{\frac{1}{4}[\sigma_Y + (1 + \eta)\sigma_{CR}]^2 - \sigma_Y\sigma_{CR}} \tag{21.45}$$

It has been found from tests on mild steel pin-ended columns that failure of an initially curved column occurs when the maximum stress in an extreme fibre reaches the yield stress, σ_Y. Also, from a wide range of tests on mild steel columns, Robertson concluded that

$$\eta = 0.003\left(\frac{L}{r}\right)$$

Substituting this value of η in Eq. (21.45) we obtain

$$\sigma = \frac{1}{2}\left[\sigma_Y + \left(1 + 0.003\frac{L}{r}\right)\sigma_{CR}\right] - \sqrt{\frac{1}{4}\left[\sigma_Y + \left(1 + 0.003\frac{L}{r}\right)\sigma_{CR}\right]^2 - \sigma_Y\sigma_{CR}} \tag{21.46}$$

In Eq. (21.46) σ_Y is a material property while σ_{CR} (from Eq. (21.25)) depends upon Young's modulus, E, and the slenderness ratio of the column. Thus Eq. (21.46) may be used to determine safe axial loads or stresses (σ) for columns of a given material in terms of the slenderness ratio. Codes of Practice tabulate maximum allowable values of average compressive stress against a range of slenderness ratios.

EXAMPLE 21.3

A column 3 m high has a rectangular thin-walled cross section 120 mm × 180 mm and is fixed at both ends; the short sides are each 6 mm thick while the long sides are each 8 mm thick. Find the safe load for the column using the Perry-Robertson formula (Eq. (21.46)) given that the yield stress in compression of mild steel is 250 N/mm^2 and the factor of safety is 3; take $E = 200000$ N/mm^2. By comparing the crippling load of the column with that given by the Euler theory deduce how the column would fail in practice.

The second moments of area about axes parallel to the sides of the column cross section are

$$I(\text{axis parallel to short sides}) = 2\left(120 \times 6 \times 90^2 + \frac{8 \times 180^3}{12}\right) = 19.4 \times 10^6 \text{mm}^4$$

$$I(\text{axis parallel to long sides}) = 2\left(180 \times 8 \times 60^2 + \frac{6 \times 120^3}{12}\right) = 12.1 \times 10^6 \text{mm}^4$$

column therefore buckles about an axis parallel to the longest sides of its cross section. The :tional area of the column is

$$A = 2(120 \times 6 + 180 \times 8) = 4320 \text{ mm}^2$$

radius of gyration is then

$$r = \sqrt{(12.1 \times 10^6/4320)} = 52.9 \text{ mm}$$

slenderness ratio is then

$$L_e/r = 0.5 \times 3 \times 10^3/52.9 = 28.4$$

ı Eq. (21.25)

$$\sigma_{CR} = \frac{\pi^2 \times 200000}{28.4^2} = 2447.3 \text{ N/mm}^2$$

:ituting the relevant values in Eq. (21.46) gives

$$\sigma = 228.5 \text{ N/mm}^2$$

safe load for the column is then

$$P = \frac{228.5 \times 4320}{3} = 329 \text{ kN}$$

Euler buckling load is $2447.3 \times 4320/10^3 = 10572$ kN

$$\frac{P(\text{Euler})}{P(\text{crippling})} = \frac{10572}{987} = 10.7$$

:olumn therefore fails by material yielding.

:ffect of cross section on the buckling of columns

nns we have considered so far have had doubly symmetrical cross sections with equal second of area about both centroidal axes. In practice, where columns frequently consist of I-section is is not the case. For example, a column having the I-section of Fig. 21.13 would buckle centroidal axis about which the flexural rigidity, *EI*, is least, i.e. *Gy*. In fact, the most efficient on from the viewpoint of instability would be a hollow circular section that has the same sec- ent of area about any centroidal axis and has as small an amount of material placed near the ›ssible. However, a disadvantage with this type of section is that connections are difficult

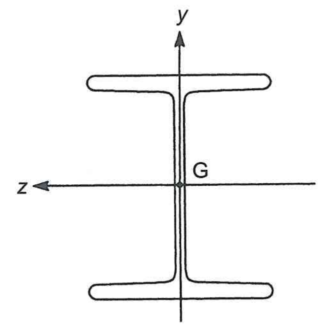


FIGURE 21.13

Effect of cross section on the buckling of columns.

In designing columns having only one cross-sectional axis of symmetry (e.g. a channel section) or none at all (i.e. an angle section having unequal legs) the least radius of gyration is taken in calculating the slenderness ratio. In the latter case the radius of gyration would be that about one of the principal axes.

Another significant factor in determining the buckling load of a column is the method of end support. We saw in Section 21.1 that considerable changes in buckling load result from changes in end conditions. Thus a column with fixed ends has a higher value of buckling load than if the ends are pinned (cf. Eqs (21.5) and (21.10)). However, we have seen that by introducing the concept of equivalent length, the buckling loads of all columns may be referred to that of a pin-ended column no matter what the end conditions. It follows that Eq. (21.46) may be used for all types of end condition, provided that the equivalent length, L_e, of the column is used. Codes of Practice list equivalent or 'effective' lengths of columns for a wide variety of end conditions. Furthermore, although a column buckles naturally in a direction perpendicular to the axis about which *EI* is least, it is possible that the column may be restrained by external means in this direction so that buckling can only take place about the other axis.

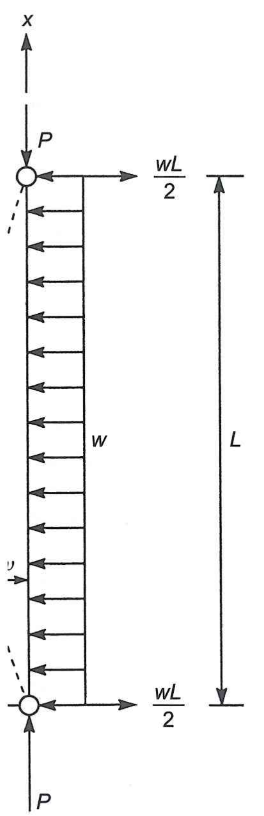
21.5 Stability of beams under transverse and axial loads

Stresses and deflections in a linearly elastic beam subjected to transverse loads as predicted by simple beam theory are directly proportional to the applied loads. This relationship is valid if the deflections are small such that the slight change in geometry produced in the loaded beam has an insignificant effect on the loads themselves. This situation changes drastically when axial loads act simultaneously with the transverse loads. The internal moments, shear forces, stresses and deflections then become dependent upon the magnitude of the deflections as well as the magnitude of the external loads. They are also sensitive, as we observed in Section 21.3, to beam imperfections such as initial curvature and eccentricity of axial loads. Beams supporting both axial and transverse loads are sometimes known as *beam-columns* or simply as *transversely loaded columns.*

We consider first the case of a pin-ended beam carrying a uniformly distributed load of intensity w and an axial load, P, as shown in Fig. 21.14. The bending moment at any section of the beam is

$$M = -Pv - \frac{wLx}{2} + \frac{wx^2}{2} = EI\frac{d^2v}{dx^2} \quad \text{(from Eq.13.3)}$$

giving



14
a uniformly loaded
mn.

$$\frac{d^2v}{dx^2} + \frac{P}{EI}v = \frac{w}{2EI}(x^2 - Lx) \tag{21.47}$$

The standard solution of Eq. (21.47) is

$$v = C_1 \cos \mu x + C_2 \sin \mu x + \frac{w}{2P}\left(x^2 - Lx - \frac{2}{\mu^2}\right)$$

where C_1 and C_2 are unknown constants and $\mu^2 = P/EI$. Substituting the boundary conditions $v = 0$ at $x = 0$ and L gives

$$C_1 = \frac{w}{\mu^2 P} \qquad C_2 = \frac{w}{\mu^2 P \sin \mu L}(1 - \cos \mu L)$$

so that the deflection is determinate for any value of w and P and is given by

$$v = \frac{w}{\mu^2 P}\left[\cos \mu x + \left(\frac{1 - \cos \mu L}{\sin \mu L}\right)\sin \mu x\right] + \frac{w}{2P}\left(x^2 - Lx - \frac{2}{\mu^2}\right) \tag{21.48}$$

In beam columns, as in beams, we are primarily interested in maximum values of stress and deflection. For this particular case the maximum deflection occurs at the centre of the beam and is, after some transformation of Eq. (21.48)

$$v_{max} = \frac{w}{\mu^2 P}\left(\sec \frac{\mu L}{2} - 1\right) - \frac{wL^2}{8P} \tag{21.49}$$

The corresponding maximum bending moment is

$$M_{max} = -Pv_{max} - \frac{wL^2}{8}$$

:q. (21.49)

$$M_{max} = \frac{w}{\mu^2}\left(1 - \sec \frac{\mu L}{2}\right) \tag{21.50}$$

ay rewrite Eq. (21.50) in terms of the Euler buckling load, $P_{CR} = \pi^2 EI/L^2$, for a pin-ended
Hence

$$M_{max} = \frac{wL^2}{\pi^2}\frac{P_{CR}}{P}\left(1 - \sec \frac{\pi}{2}\sqrt{\frac{P}{P_{CR}}}\right) \tag{21.51}$$

pproaches P_{CR} the bending moment (and deflection) becomes infinite. However, the above the-
d on the assumption of small deflections (otherwise d^2v/dx^2 would not be a close approximation
ıre) so that such a deduction is invalid. The indication is, though, that large deflections will be
ɔy the presence of a compressive axial load no matter how small the transverse load might be.
consider now the beam column of Fig. 21.15 with pinned ends carrying a concentrated load
tance a from the upper support.

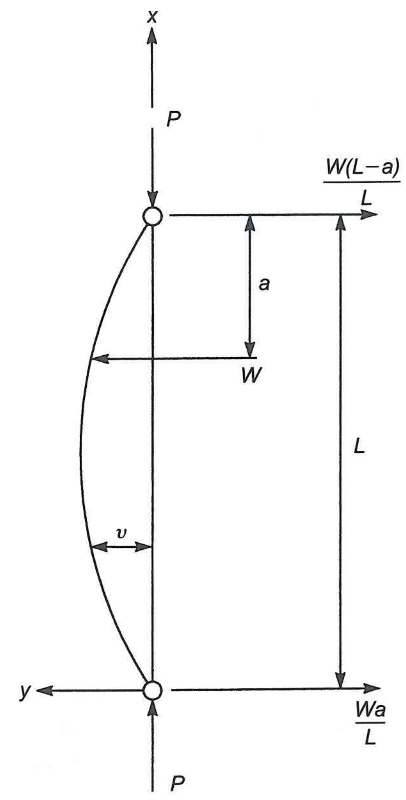


FIGURE 21.15

Beam-column supporting a point load.

$$EI\frac{d^2v}{dx^2} = M = -Pv - \frac{Wa}{L} \tag{21.52}$$

and for $x \geq L - a$,

$$EI\frac{d^2v}{dx^2} = M = -Pv - \frac{W}{L}(L - a)(L - x) \tag{21.53}$$

Writing

$$\mu^2 = \frac{P}{EI}$$

Equation (21.52) becomes

$$\frac{d^2v}{dx^2} + \mu^2 v = -\frac{Wa}{EIL}x$$

the general solution of which is

$$v = C_1 \cos \mu x + C_2 \sin \mu x - \frac{Wa}{PL}x \tag{21.54}$$

Similarly the general solution of Eq. (21.53) is

$$v = C_3 \cos \mu x + C_4 \sin \mu x - \frac{W}{PL}(L - a)(L - x) \tag{21.55}$$

where C_1, C_2, C_3 and C_4 are constants which are found from the boundary conditions as follows.

When $x = 0$, $v = 0$, therefore from Eq. (21.54) $C_1 = 0$. At $x = L$, $v = 0$ giving, from Eq. (21.55), $C_3 = -C_4 \tan \mu L$. At the point of application of the load the deflection and slope of the beam given by Eqs (21.54) and (21.55) must be the same. Hence equating deflections

$$C_2 \sin \mu(L - a) - \frac{Wa}{PL}(L - a) = C_4[\sin \mu(L - a) - \tan \mu L \cos \mu(L - a)] - \frac{Wa}{PL}(L - a)$$

and equating slopes

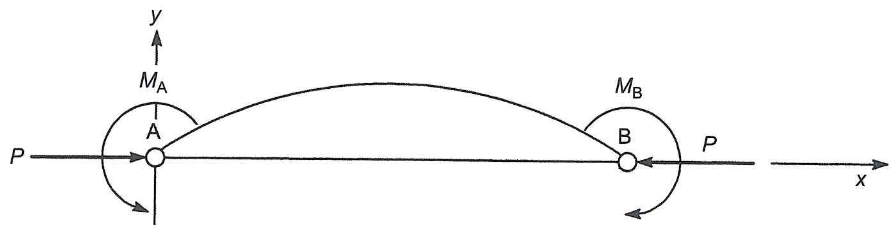
$$C_2\mu \cos \mu(L - a) - \frac{Wa}{PL} = C_4\mu[\cos \mu(L - a) + \tan \mu L \sin \mu(L - a)] + \frac{Wa}{PL}(L - a)$$

Solving the above equations for C_2 and C_4 and substituting for C_1, C_2, C_3 and C_4 in Eqs (21.54) and (21.55) we have

$$v = \frac{W \sin \mu a}{P\mu \sin \mu L}\sin \mu x - \frac{Wa}{PL}x \text{ for } x \leq L - a \tag{21.56}$$

$$v = \frac{W \sin \mu(L - a)}{P\mu \sin \mu L}\sin \mu(L - x) - \frac{Wa}{PL}(L - a)(L - x) \text{ for } x \geq L - a \tag{21.57}$$

These equations for the beam-column deflection enable the bending moment and resulting bending stresses to be found at all sections.



.16
ımn supporting end moments.

:icular case arises when the load is applied at the centre of the span. The deflection curve is ɲetrical with a maximum deflection under the load of

$$v_{max} = \frac{W}{2P\mu}\tan\frac{\mu L}{2} - \frac{WL}{4P}$$

ʳ we consider a beam column subjected to end moments, M_A and M_B, in addition to an axial ʲig. 21.16). The deflected form of the beam column may be found by using the principle of :ion and the results of the previous case. First we imagine that M_B acts alone with the axial ˙we assume that the point load, W, moves towards B and simultaneously increases so that the ʼa = constant = M_B then, in the limit as a tends to zero, we have the moment M_B applied at B. :tion curve is then obtained from Eq. (21.56) by substituting μa for sin μa (since μa is now) and M_B for W_a. Thus

$$v = \frac{M_B}{P}\left(\frac{\sin \mu x}{\sin \mu L} - \frac{x}{L}\right) \tag{21.58}$$

ıd the deflection curve corresponding to M_A acting alone in a similar way. Suppose that W ʳards A such that the product $W(L - a)$ = constant = M_A. Then as $(L - a)$ tends to zero we $(L - a) = \mu(L - a)$ and Eq. (21.57) becomes

$$v = \frac{M_A}{P}\left[\frac{\sin \mu(L-x)}{\sin \mu L} - \frac{(L-x)}{L}\right] \tag{21.59}$$

fect of the two moments acting simultaneously is obtained by superposition of the results of ꜟ) and (21.59). Hence, for the beam-column of Fig. 21.16

$$v = \frac{M_B}{P}\left(\frac{\sin \mu x}{\sin \mu L} - \frac{x}{L}\right) + \frac{M_A}{P}\left[\frac{\sin \mu(L-x)}{\sin \mu L} - \frac{(L-x)}{L}\right] \tag{21.60}$$

ɔn (21.60) is also the deflected form of a beam-column supporting eccentrically applied end and B. For example, if e_A and e_B are the eccentricities of P at the ends A and B, respectively, : Pe_A, $M_B = Pe_B$, giving a deflected form of

$$v = e_B\left(\frac{\sin \mu x}{\sin \mu L} - \frac{x}{L}\right) + e_A\left[\frac{\sin \mu(L-x)}{\sin \mu L} - \frac{(L-x)}{L}\right] \tag{21.61}$$

ɔeam-column configurations featuring a variety of end conditions and loading regimes may

EXAMPLE 21.4

The pin-jointed column shown in Fig. 21.17 carries a compressive load P applied eccentrically at a distance e from the axis of the column. Determine the maximum bending moment in the column.

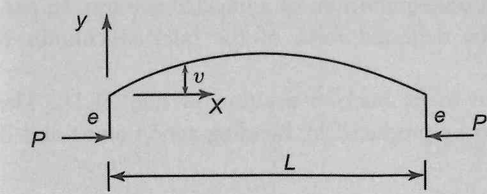


FIGURE 21.17

Eccentrically loaded column of Ex. 21.4.

The bending moment at any section of the column is given by

$$M = P(e + v)$$

Then, by comparison with Eq. 21.1

$$EI\frac{d^2v}{dx^2} = -P(e + v)$$

giving

$$\frac{d^2v}{dx^2} + \mu^2 v = -\frac{Pe}{EI} \quad (\mu^2 = P/EI)$$

the solution of which is

$$v = A\cos\mu x + B\sin\mu x - e$$

The boundary conditions are $v = 0$ when $x = 0$ and $(dv/dx) = 0$ when $x = L/2$. From the first of these $A = e$ while from the second

$$B = e\tan\frac{\mu L}{2}$$

The equation for the deflected shape of the column is then

$$v = e\left[\frac{(\cos\mu(x - L/2)}{\cos(\mu L/2)} - 1\right]$$

The maximum value of v occurs at mid-span where $x = L/2$, that is

$$v_{max} = e\left[\sec\left(\frac{\mu L}{2}\right) - 1\right]$$

The maximum bending moment is given by

$$M_{max} = Pe + Pv_{max}$$

so that

$$M_{max} = Pe\sec\left(\frac{\mu L}{2}\right)$$

:nergy method for the calculation of buckling loads in columns Rayleigh–Ritz Method)

:hat the total potential energy of an elastic body possesses a stationary value in an equilibrium Section 15.3) may be used to investigate the neutral equilibrium of a buckled column. In par- e energy method is extremely useful when the deflected form of the buckled column is and has to be 'guessed'.

ve shall consider the pin-ended column shown in its buckled position in Fig. 21.18. The r strain energy, U, of the column is assumed to be produced by bending action alone and is Eq. (9.21), i.e.

$$U = \int_0^L \frac{M^2}{2EI}\,dx \tag{21.62}$$

tively, since $EI\,d^2v/dx^2 = M$ (Eq. (13.3))

$$U = \frac{EI}{2}\int_0^L \left(\frac{d^2v}{dx^2}\right)^2 dx \tag{21.63}$$

otential energy, V, of the buckling load, P_{CR}, referred to the straight position of the column is then

$$V = -P_{CR}\delta$$

; the axial movement of P_{CR} caused by the bending of the column from its initially straight From Fig. 21.18 the length δL in the buckled column is

$$\delta L = (\delta x^2 + \delta v^2)^{1/2}$$

dv/dx is small then

$$\delta L \simeq \delta x\left[1 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right]$$

$$L = \int_0^{L'} \left[1 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right] dx$$

$$L = L' + \int_0^{L'} \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

ore

$$\delta = L - L' = \int_0^{L'} \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

$$\int_0^{L'} \frac{1}{2}\left(\frac{dv}{dx}\right)^2$$

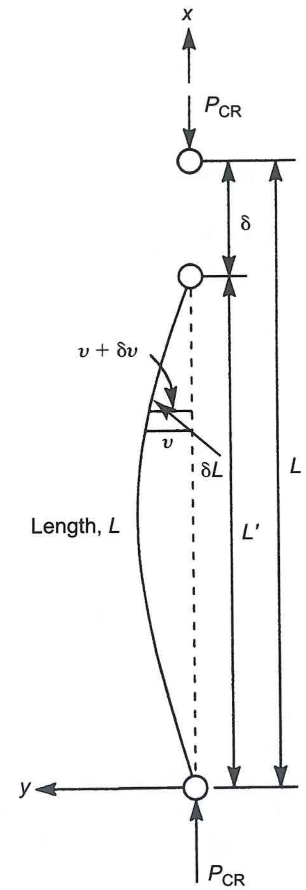


FIGURE 21.18

Shortening of a column due to buckling.

only differs from

$$\int_0^L \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

by a term of negligible order, we write

$$\delta = \int_0^L \frac{1}{2}\left(\frac{dv}{dx}\right)^2 dx$$

giving

$$V = -\frac{P_{CR}}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx \tag{21.64}$$

The total potential energy of the column in the neutral equilibrium of its buckled state is therefore

$$U + V = \int_0^L \frac{M^2}{2EI}\,dx - \frac{P_{CR}}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx \tag{21.65}$$

or, using the alternative form of U from Eq. (21.63)

$$U + V = \frac{EI}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx - \frac{P_{CR}}{2}\int_0^L \left(\frac{dv}{dx}\right)^2 dx \tag{21.66}$$

We shall now assume a deflected shape having the equation

$$v = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} \tag{21.67}$$

This satisfies the boundary conditions of

$$(v)_{x=0} = (v)_{x=L} = 0 \quad \left(\frac{d^2v}{dx^2}\right)_{x=0} = \left(\frac{d^2v}{dx^2}\right)_{x=L} = 0$$

and is capable, within the limits for which it is valid and if suitable values for the constant coefficients, A_n, are chosen, of representing any continuous curve. We are therefore in a position to find P_{CR} exactly. Substituting Eq. (21.67) into Eq. (21.66) gives

$$U + V = \frac{EI}{2}\int_0^L \left(\frac{\pi}{L}\right)^4 \left(\sum_{n=1}^{\infty} n^2 A_n \sin\frac{n\pi x}{L}\right)^2 dx - \frac{P_{CR}}{2}\int_0^L \left(\frac{\pi}{L}\right)^2 \left(\sum_{n=1}^{\infty} nA_n \cos\frac{n\pi x}{L}\right)^2 dx \tag{21.68}$$

The product terms in both integrals of Eq. (21.68) disappear on integration leaving only integrated values of the squared terms. Thus

$$U + V = \frac{\pi^4 EI}{4L^3}\sum_{n=1}^{\infty} n^4 A_n^2 - \frac{\pi^2 P_{CR}}{4L}\sum_{n=1}^{\infty} n^2 A_n^2 \tag{21.69}$$

Assigning a stationary value to the total potential energy of Eq. (21.69) with respect to each coefficient, A_n, in turn, then taking A_n as being typical, we have ich

$$P_{CR} = \frac{\pi^2 E I n^2}{L^2}$$

e that each term in Eq. (21.67) represents a particular deflected shape with a corresponding criti- Hence the first term represents the deflection of the column shown in Fig. 21.18 with EI/L^2. The second and third terms correspond to the shapes shown in Fig. 21.4(b) and (c) having ads of $4\pi^2 EI/L^2$ and $9\pi^2 EI/L^2$ and so on. Clearly the column must be constrained to buckle into re complex forms. In other words, the column is being forced into an unnatural shape, is conse- iffer and offers greater resistance to buckling, as we observe from the higher values of critical load. deflected shape of the column is known, it is immaterial which of Eqs. (21.65) or (21.66) is he total potential energy. However, when only an approximate solution is possible, Eq. (21.65) ole since the integral involving bending moment depends upon the accuracy of the assumed form reas the corresponding term in Eq. (21.66) depends upon the accuracy of d^2v/dx^2. Generally, for ed deflection curve v is obtained much more accurately than d^2v/dx^2.

se that the deflection curve of a particular column is unknown or extremely complicated. We then reasonable shape which satisfies as far as possible the end conditions of the column and the pattern lected shape (Rayleigh–Ritz method). Generally the assumed shape is in the form of a finite series a series of unknown constants and assumed functions of x. Let us suppose that v is given by

$$v = A_1 f_1(x) + A_2 f_2(x) + A_3 f_3(x)$$

tution in Eq. (21.65) results in an expression for total potential energy in terms of the critical the coefficients A_1, A_2 and A_3 as the unknowns. Assigning stationary values to the total potential th respect to A_1, A_2 and A_3 in turn produces three simultaneous equations from which the A_2, A_1/A_3 and the critical load are determined. Absolute values of the coefficients are unobtain- the displacements of the column in its buckled state of neutral equilibrium are indeterminate.

LE 21.5

energy method to determine the buckling load of the column shown in Fig. 21.19.

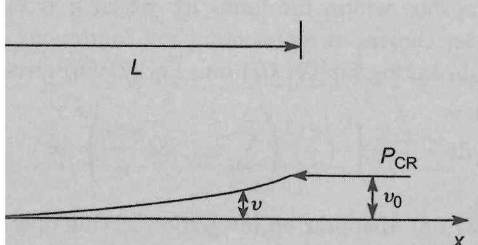


FIGURE 21.19

Buckling load for a built-in column by the energy method.

pproximate shape may be deduced from the deflected shape of a cantilever loaded at its free is, from Eq. (iv) of Ex. 13.1

$$v = \frac{v_0 x^2}{2L^3}(3L - x)$$

This expression satisfies the end conditions of deflection, viz. $v = 0$ at $x = 0$ and $v = v_0$ at $x = L$. In addition, it satisfies the conditions that the slope of the column is zero at the built-in end and that the bending moment, i.e. d^2v/dx^2, is zero at the free end. The bending moment at any section is $M = P_{CR}(v_0 - v)$ so that substitution for M and v in Eq. (21.65) gives

$$U + V = \frac{P_{CR}^2 v_0^2}{2EI}\int_0^L \left(1 - \frac{3x^2}{2L^2} + \frac{x^3}{2L^3}\right)^2 dx - \frac{P_{CR}}{2}\int_0^L \left(\frac{3v_0}{2L^3}\right)^2 x^2(2L - x)^2 dx$$

Integrating and substituting the limits we have

$$U + V = \frac{17}{35}\frac{P_{CR}^2 v_0^2 L}{2EI} - \frac{3}{5}P_{CR}\frac{v_0^2}{L}$$

Hence

$$\frac{\partial(U + V)}{\partial v_0} = \frac{17}{35}\frac{P_{CR}^2 v_0 L}{EI} - \frac{6P_{CR}\, v_0}{5L} = 0$$

from which

$$P_{CR} = \frac{42EI}{17L^2} = 2.471\frac{EI}{L^2}$$

This value of critical load compares with the exact value (see Eq. (21.15)) of $\pi^2 EI/4L^2 = 2.467\, EI/L^2$; the error, in this case, is seen to be extremely small. Approximate values of critical load obtained by the energy method are always greater than the correct values. The explanation lies in the fact that an assumed deflected shape implies the application of constraints in order to force the column to take up an artificial shape. This, as we have seen, has the effect of stiffening the column with a consequent increase in critical load.

It will be observed that the solution for the above example may be obtained by simply equating the increase in internal energy *(U)* to the work done by the external critical load $(-V)$. This is always the case when the assumed deflected shape contains a single unknown coefficient, such as v_0, in the above example.

In this chapter we have investigated structural instability with reference to the overall buckling or failure of columns subjected to axial load and also to bending. The reader should also be aware that other forms of instability occur. For example, the compression flange in an I-section plate girder can buckle laterally when the girder is subjected to bending moments unless it is restrained. Furthermore, thin-walled open section beams that are weak in torsion can exhibit torsional instability, i.e. they suddenly twist, when subjected to axial load. These forms of instability are considered in more advanced texts.

PROBLEMS

P.21.1 A uniform column of length L and flexural rigidity EI is built-in at one end and is free at the other. It is designed so that its lowest buckling load is P (Fig. P.21.1(a)). Subsequently it is required to carry an increased load and for that it is provided with a lateral spring at the free end (Fig. P.21.1 (b)). Determine the necessary spring stiffness, k, so that the buckling load is $4P$.

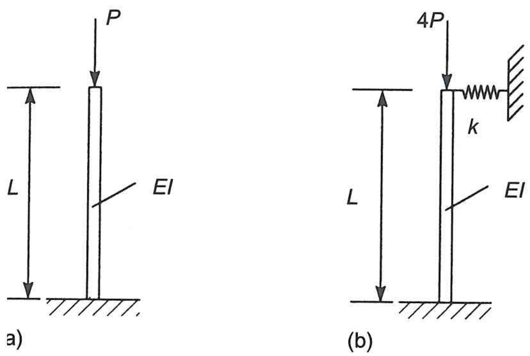


FIGURE P.21.1

pin-ended column of length L and flexural rigidity EI is reinforced to give a flexural rigidity I over its central half. Determine its lowest buckling load.

ıs. $24.2EI/L^2$.

uniform pin-ended column of length L and flexural rigidity EI has an initial curvature such at the lateral displacement at any point between the column and the straight line joining its ds is given by

$$v_0 = a\frac{4x}{L^2}(L-x)$$

ıere a is the initial displacement at the mid-length of the column and the origin for x is at .e end.

Show that the maximum bending moment due to a compressive axial load, P, is given by

$$M_{\max} = -\frac{8aP}{(\mu L)^2}\left(\sec\frac{\mu L}{2} - 1\right) \quad \text{where } \mu^2 = \frac{P}{EI}$$

compression member is made of circular section tube having a diameter d and thickness t d is curved initially so that its initial deflected shape may be represented by the expression

$$v_0 = \delta\sin\left(\frac{\pi x}{L}\right)$$

which δ is the displacement at its mid-length and the origin for x is at one end.

Show that if the ends are pinned, a compressive load, P, induces a maximum direct stress, ıax, given by

$$\sigma_{\max} = \frac{P}{\pi dt}\left(1 + \frac{1}{1-\alpha}\frac{4\delta}{d}\right)$$

.ere $\alpha = P/P_{CR}$ and $P_{CR} = \pi^2 EI/L^2$. Assume that t is small compared with d so that the ›ss-sectional area of the tube is πdt and its second moment of area is $\pi d^3 t/8$.

.e uniform pin-ended column shown in Fig. P.21.5 is bent at the centre so that the :entricity there is δ. If the two halves of the column are otherwise straight and have a

flexural stiffness EI find the maximum bending moment when the column carries a compressive load P.

Ans.

$$-P\frac{2\delta}{L}\sqrt{(EI/P)}\tan\sqrt{(P/EI)}\frac{L}{2}$$

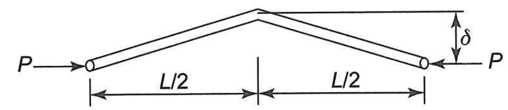


FIGURE P.21.5

P.21.6 A straight uniform column of length L and bending stiffness EI is subjected to uniform lateral loading w/unit length. The end attachments do not restrict rotation of the column ends. The longitudinal compressive force P has eccentricity e from the centroids of the end sections and is placed so as to oppose the bending effect of the lateral loading as shown in Fig. P.21.6. The eccentricity e can be varied and is to be adjusted to the value which, for given values of P and w, will result in the least maximum bending moment on the column. Show that

$$e = (w/P\mu^2)\tan^2(\mu L/4)$$

where $\mu^2 = P/EI$. Also deduce the end moment that gives the optimum condition when P tends to zero.

Ans. $wL^2/16$.

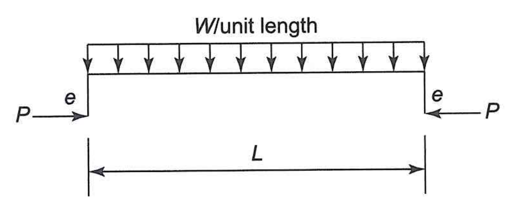


FIGURE P.21.6

P.21.7 A rectangular portal frame ABCD is rigidly fixed to foundations A and D and is subjected to a compression load P as shown in Fig. P.21.7. If all the members have the same bending stiffness EI show that the buckling loads for modes which are symmetrical about the vertical centre line are given by the transcendental equation

$$\frac{\mu a}{2} = -\frac{1}{2}\left(\frac{a}{b}\right)\tan\left(\frac{\mu a}{2}\right)$$

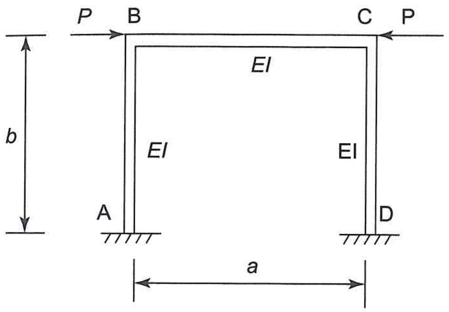


FIGURE P.21.7

n the experimental determination of the buckling loads for 12.5 mm diameter, mild steel, in-ended columns, two of the values obtained were:

(i) length 500 mm, load 9800 N,
(ii) length 200 mm, load 26 400 N.
(a) Determine whether either of these values conforms to the Euler theory for buckling load.
(b) Assuming that both values are in agreement with the Rankine formula, find the constants σ_s and k. Take $E = 200\ 000\ \text{N/mm}^2$.

ns. (a) (i) conforms with Euler theory.
(b) $\sigma_s = 317\ \text{N/mm}^2$ $k = 1.16 \times 10^{-4}$.

. tubular column has an effective length of 2.5 m and is to be designed to carry a safe load of 00 kN. Assuming an approximate ratio of thickness to external diameter of 1/16, determine practical diameter and thickness using the Rankine formula with $\sigma_s = 330\ \text{N/mm}^2$ and $= 1/7500$. Use a safety factor of 3.

ns. Diameter = 128 mm thickness = 8 mm.

. short length of hollow tube 32 mm external diameter and 25 mm internal diameter yielded ı a compression test at a load of 70 kN. When a 2.5 m length of the same tube was tested as column with fixed ends the failure load was 24.1 kN. Assuming that σ_S in the Rankine ›rmula is given by the first test find the value of the constant k and hence the crippling load ›r a column 1.5 m in length when used as a column with pinned ends.

ns. $k = 0.000126$, 18.7 kN.

mild steel column is 6 m long, is fixed at both ends and has the cross section shown in ig. P.21.11. Given that the yield stress in compression of mild steel is $300\ \text{N/mm}^2$ calculate ıe maximum allowable load for the column using the Perry-Robertson formula. Take $= 200000\ \text{N/mm}^2$ and assume a factor of safety of 2.

ns. 406.8 kN.

column is fabricated from two 305 mm × 305 mm × 158 kg Universal Column sections aced side by side as shown in Fig. P.21.12. The column has fixed ends and an overall height 12 m. Given that the yield stress in compression of mild steel is $250\ \text{N/mm}^2$ calculate the ıaximum allowable load for the column using the Perry-Robertson formula. Take $= 200000\ \text{N/mm}^2$ and assume a factor of safety of 2. The properties of a *single*)5 mm × 305 mm × 158 kg UC are:

Area = $201.2\ \text{cm}^2$, $I_z = 38740\ \text{cm}^4$, $I_y = 12524\ \text{cm}^4$

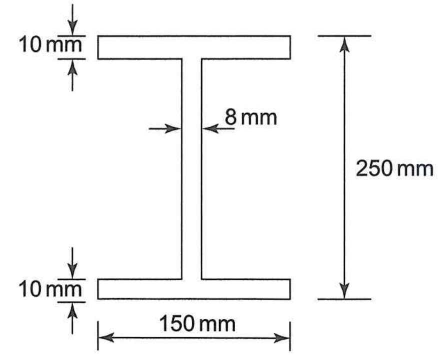


FIGURE P.21.11

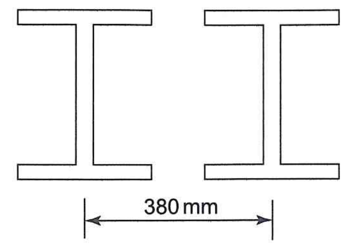


FIGURE P.21.12

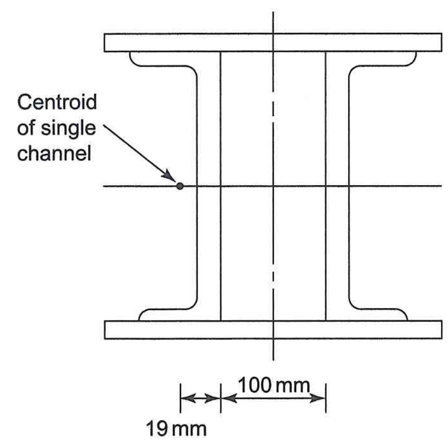


FIGURE P.21.13

P.21.13 A column is fabricated from two 250 mm × 75 mm channel sections attached to two flange plates each 300 mm × 12 mm as shown in Fig. P.21.13; the column height is 10 m and the ends of the column are fixed. If the yield stress of mild steel in compression is $250\ \text{N/mm}^2$ use the Perry-Robertson formula to calculate the crippling load for the column. Take $E = 205000\ \text{N/mm}^2$. The properties of a *single* 250 mm × 75 mm channel are:

Area = $36.6\ \text{cm}^2$, $I_z = 3440\ \text{cm}^4$, $I_y = 166\ \text{cm}^4$.

ıild steel pin-ended column is 2.5 m long and has the cross section shown in Fig. P.21.14. ıe yield stress in compression of mild steel is 300 N/mm^2, determine the maximum load column can withstand using the Robertson formula. Compare this value with that licted by the Euler theory.

576 kN, P (Robertson)/P (Euler) = 0.62.

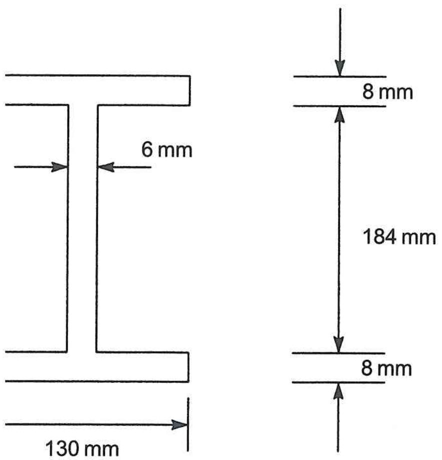


FIGURE P.21.14

n-ended column of length L has its central portion reinforced, the second moment of its being I_2 while that of the end portions, each of length a, is I_1. Use the Rayleigh–Ritz ıod to determine the critical load of the column assuming that its centreline deflects into ɔarabola $v = kx(L - x)$ and taking the more accurate of the two expressions for bending ıent.

n the case where $I_2 = 1.6I_1$ and $a = 0.2\,L$ find the percentage increase in strength due to ːeinforcement.

$P_{CR} = 14.96EI_1/L^2$, 52%.

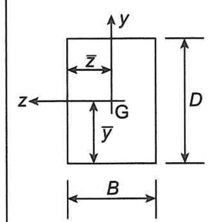
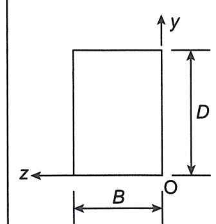
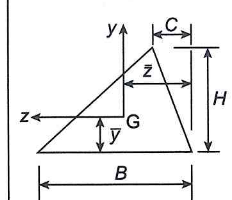
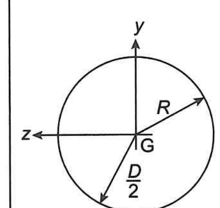
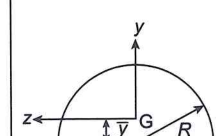
bular column of length L is tapered in wall thickness so that the area and the second ıent of area of its cross section decrease uniformly from A_1 and I_1 at its centre to $0.2A_1$ $0.2I_1$ at its ends, respectively.

Ɩssuming a deflected centreline of parabolic form and taking the more correct form for ɔending moment, use the Rayleigh–Ritz method to estimate its critical load; the ends of ːolumn may be taken as pinned. Hence show that the saving in weight by using such a mn instead of one having the same radius of gyration and constant thickness is ıt 15%.

$7EI_1/L^2$.

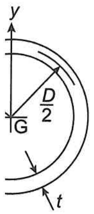
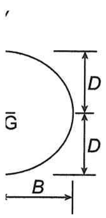
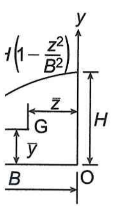
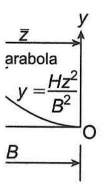
Appendix A: Table of Section Properties

Table A.1

Section	A	$\bar{z}$	$\bar{y}$	I_z	I_y	I_{zy}
	BD	$\frac{B}{2}$	$\frac{D}{2}$	$\frac{BD^3}{12}$	$\frac{DB^3}{12}$	0
				$\frac{BD^3}{3}$	$\frac{DB^3}{3}$	$\frac{B^2D^2}{4}$
	$\frac{BH}{2}$	$\frac{B+C}{3}$	$\frac{H}{3}$	$\frac{BH^3}{36}$	$\frac{BH}{36}(B^2 - BC + C^2)$	$\frac{BH^2}{72}(B - 2C)$
	$\pi R^2, \frac{\pi D^2}{4}$			$\frac{\pi R^4}{4}, \frac{\pi D^4}{64}$	$\frac{\pi R^4}{4}, \frac{\pi D^4}{64}$	0
	$\frac{\pi R^2}{2}$		$\frac{4R}{3\pi}$	$\approx 0.11R^4$	$\frac{\pi R^4}{8}$	0

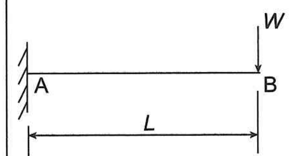
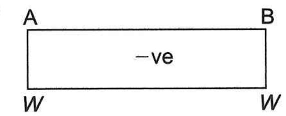
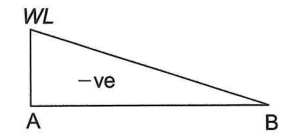
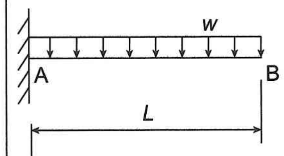
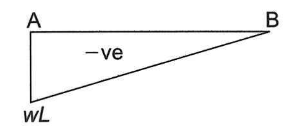
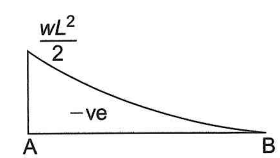
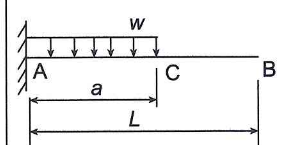
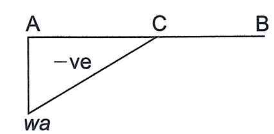
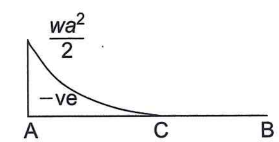
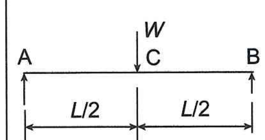
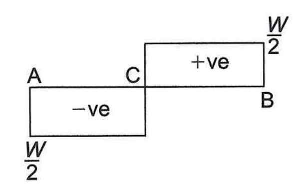
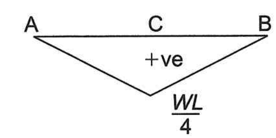
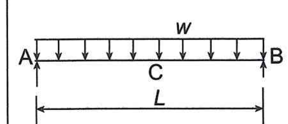
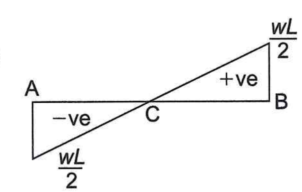
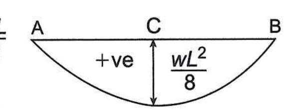
(Continued)

(Continued)

	A	$\bar{z}$	$\bar{y}$	I_z	I_y	I_{zy}
	$2\pi Rt,\ \pi Dt$			$\pi R^3 t, \frac{\pi D^3 t}{8}$	$\pi R^3 t, \frac{\pi D^3 t}{8}$	0
	πBD			$\frac{\pi BD^3}{4}$	$\frac{\pi BD^3}{4}$	0
	$\frac{2BH}{3}$	$\frac{3B}{8}$	$\frac{2H}{5}$			
	$\frac{BH}{3}$	$\frac{3B}{4}$	$\frac{3H}{10}$			

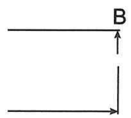
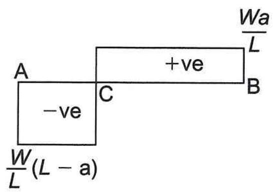
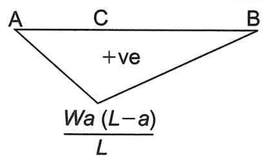
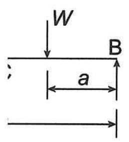
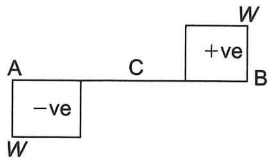
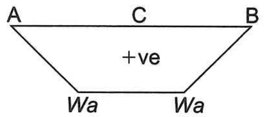
Appendix B: Bending of Beams: Standard Cases

Table B.1

Beam	SF distribution	BM distribution	DEF
			$\frac{WL^3}{3EI}$ (B) (max)
			$\frac{wL^4}{8EI}$ (B) (max)
			$\frac{wa^3}{24EI}(4L-a)$ (max) (B)
			$\frac{WL^3}{48EI}$ (C) (max)
			$\frac{5wL^4}{384EI}$ (C) (max)

(Continued)

ontinued)

	SF distribution	BM distribution	DEF
			$\frac{wa^2(a-L)^2}{3EIL}$ (C) (not max)
			$\frac{WL^3}{6EI}\left[\frac{3a}{4L}-\left(\frac{a}{L}\right)^3\right]$ (C) (max)

Index

Note: Page number followed by "*f*", "*t*", and "*b*" refers to figures, tables, and boxes respectively.

A

B

C

T

W

Y